Solutions to some additional M2P1 exercises

Exercise. Let f be the function defined as follows:

$$f(x) = \begin{cases} x, & x < 0, \\ 1 + x, & x \geq 0. \end{cases}$$

Prove in two different ways that f is not continuous at 0.

Solution.
Method 1. To get a contradiction, let us suppose that f is continuous at 0. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that $|x| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$. Let us take $\epsilon = \frac{1}{2}$. Then there exists a $\delta > 0$ such that $|x| < \delta \Rightarrow |f(x) - 1| < \frac{1}{2}$. However, for any $\delta > 0$, setting $x = -\frac{\delta}{2}$, we find that $0 < \frac{\delta}{2} = |x| < \delta$ but $|f(x) - 1| = 1 + \frac{\delta}{2} \geq \frac{1}{2}$. Contradiction. Hence f is discontinuous at 0.
Method 2. We can also use the theorem linking continuity with sequences. Let (x_n) and (y_n) be sequences such that $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$, $n \in \mathbb{N}$. Then $f(x_n) = 1 + \frac{1}{n} \to 1$ whereas $f(y_n) = -\frac{1}{n} \to 0$. Hence f is not continuous at 0 by Theorem 6.5 (M1P1).

Exercise. Define $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x) = x_1 x_2$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Prove (directly from the definition of continuity) that this function is continuous at all $a = (a_1, a_2) \in \mathbb{R}^2$.

Solution. Let $a \in \mathbb{R}^2$. Fix $\epsilon > 0$. Observe that inequalities $|x_1 - a_1| \leq |x - a|$ and $|x_2 - a_2| \leq |x - a|$ are obviously valid. Then we observe

$$|f(x) - f(a)| = |x_1x_2 - a_1a_2| = |x_1x_2 - a_1x_2 + a_1x_2 - a_1a_2| \leq |x_2||x_1 - a_1| + |a_1||x_2 - a_2| \leq$$
$$(|x_2| + |a_1|)|x - a| \leq (|x_2 - a_2| + |a_2| + |a_1|)|x - a| \leq |x - a|^2 + (|a_1| + |a_2|)|x - a|.$$

Now we can use two methods to choose δ, dependent on the taste.
Method 1. Let $\delta = \min\{1, \epsilon/(1 + |a_1| + |a_2|)\}$. Let $|x - a| < \delta$. Then $|x_2| \leq |x_2 - a_2| + |a_2| \leq |x - a| + |a_2| < 1 + |a_2|$, so by the above estimates we can conclude $|f(x) - f(a)| \leq (|x_2| + |a_1|)|x - a| < (1 + |a_2| + |a_1|)\delta \leq \epsilon$.

Method 2. We can use $|x-a|^2 + (|a_1| + |a_2|)|x-a| < \epsilon \Leftrightarrow |x-a| + \frac{|a_1|+|a_2|}{2} < \left[\epsilon + \left(\frac{|a_1|+|a_2|}{2}\right)^2\right]^{\frac{1}{2}}$, which is equivalent to $|x-a| < \left[\epsilon + \left(\frac{|a_1|+|a_2|}{2}\right)^2\right]^{\frac{1}{2}} - \frac{|a_1|+|a_2|}{2}$. So, if we take $\delta = \left[\epsilon + \left(\frac{|a_1|+|a_2|}{2}\right)^2\right]^{\frac{1}{2}} - \frac{|a_1|+|a_2|}{2}$, then $\delta > 0$ and $|f(x) - f(a)| \leq |x - a|^2 + (|a_1| + |a_2|)|x - a| < \epsilon$ by the estimates above. Hence f is continuous at $a \in \mathbb{R}^2$.

Exercise. Define $f : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}$ by $f(x) = \frac{x_1}{|x|}$ for all $x \in \mathbb{R}^2 \backslash \{0\}$. Does f have a limit as $x \to 0$? Justify your answer.

Solution. This function f has no limit as $x \to 0$. Indeed, let us take sequences (z_n) and (y_n) defined by $z_n = (\frac{1}{n}, 0)$ and $y_n = (0, \frac{1}{n})$. Both sequences converge to $(0, 0)$ but $z_n \neq (0, 0)$, $y_n \neq (0, 0)$ for all n. Now, for all $n \geq 1$, $f(z_n) = 1 \to 1$, whereas $f(y_n) = 0 \to 0$. Thus by the Theorem 6.5 (M1P1) and Theorem 1.3 we conclude that there is no $l \in \mathbb{R}^2$ such that $f(x) \to l$ as $x \to 0$.

Exercise. Suppose that f is a function such that $f'(x) = 0$ for all $x \in \mathbb{R}$. Prove that f must be a constant function.

Solution. Since f is differentiable everywhere, it is also continuous everywhere by Theorem 3.1. Let $a < b$ be two points in $\mathbb{R}$. Then f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value

Theorem 4.3, there exists a $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. But by assumption we must have $f'(c) = 0$. This implies $f(a) = f(b)$. Since the choice of a and b was arbitrary, it follows that f is constant.

Theorem. Suppose that f is continuous on a closed interval $[a,b]$. Then f is bounded (both above and below) on $[a,b]$.

Proof. Assume that f is not bounded above on $[a,b]$. Then for all real numbers M there exists $x \in [a,b]$ such that $f(x) > M$. In particular, for every $n \in \mathbb{N}$ there exists some $x_n \in [a,b]$ such that $f(x_n) > n$. Since $a \leq x_n \leq b$, sequence (x_n) is bounded and by the Bolzano-Weierstrass Theorem has a convergent subsequence, say (u_m), converging to some u. By the basic property of closed intervals it follows that $u \in [a,b]$. Since $u_m \to u$ and f is continuous on $[a,b]$, hence also at u, it follows that $f(u_m) \to f(u)$ as $m \to \infty$. Since (u_m) is a subsequence of (x_n), we have $u_m = x_n$ for some index $n \geq m$. It follows that $f(u_m) = f(x_n) > n \geq m$. Hence $f(u_m) > m$ for all $m \in \mathbb{N}$. This contradicts $f(u_m) \to f(u)$ as $m \to \infty$. Indeed, if we take any $m \in \mathbb{N}$ such that $m \geq f(u)+1$ we see that $f(u_m) > m \geq f(u)+1$, so that $(f(u_m))$ can not converge to $f(u)$ as $m \to \infty$. Repeating the proof for $-f$ we find that $-f$ is bounded above, so f is also bounded below. This completes the proof.

Exercise. State true or false with reasoning:
(i) If f is bounded on $[a,b]$ then f is continuous on $[a,b]$.
(ii) If f is continuous on (a,b) then f is bounded on (a,b).

Solution. (i) False. Consider f defined by

$$f(x) = \begin{cases} 1, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

Then f is bounded on $[0,2]$ (e.g. by 2) but it is not continuous on the interval $[0,2]$ since it is discontinuous at 1.
(ii) False. Consider f such that $f(x) = \frac{1}{x}$ on the interval $(0,1)$. Then f is continuous at all $a \in (0,1)$ by Theorem 1.3. However, f is not bounded on $(0,1)$. Indeed, let $A > 0$. Let $\delta = \frac{1}{A} > 0$. Then $0 < x < \delta$ implies $x < \delta = \frac{1}{A}$, so $f(x) = \frac{1}{x} > A$. Hence f is not bounded above (and hence not bounded) on the interval $(0,1)$, even though it is continuous on $(0,1)$.

Exercise. Let $g(x) \to m$ as $x \to a$ and $f(y) \to l$ as $y \to m$. Show that this does not imply $f(g(x)) \to l$ as $x \to a$.

Solution. Let $g(x) = 0$ for all x. Let $a = 0$. Then $g(x) \to m = 0$ as $x \to a = 0$. Define

$$f(y) = \begin{cases} 1, & y \neq 0, \\ 0, & y = 0 \end{cases}$$

Then $f(y) \to l = 1$ as $y \to m = 0$ but $f(g(x)) = f(0) = 0 \to 0 \neq 1$ as $x \to a = 0$.