M2P1 (2004) Dr M Ruzhansky
Short list of statements.

Chapter 1: Limits and continuity.

Def 1.1 (*limits*) Suppose f is a function defined for all x in some interval containing a point a, except (perhaps) at a itself. Then "$f(x) \to l$ as $x \to a$" means that given any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$. Here and in what follows l should be a finite number.

Similarly, we can define right and left limits. So, "$f(x) \to l$ as $x \to a+$" means $\forall \epsilon > 0\ \exists \delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - l| < \epsilon$, and "$f(x) \to l$ as $x \to a-$" means $\forall \epsilon > 0$ $\exists \delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - l| < \epsilon$, respectively.

Relation to continuity in M1P1: "f is continuous (left cts, right cts) at a" means that (i) $f(a)$ is defined, and (ii) $f(x) \to f(a)$ as $x \to a$ ($x \to a-$, $x \to a+$, resp.)

Basic rules for limits: Suppose $f(x) \to l$ and $g(x) \to m$ as $x \to a$. Then:
(1.2) $f(x) + g(x) \to l + m$ as $x \to a$.
(1.3) $f(x)g(x) \to lm$ as $x \to a$.
(1.4) $f(x)/g(x) \to l/m$ as $x \to a$, provided $m \neq 0$.

Connection between limits of sequences and of functions:
(1.5) Suppose $f(x) \to l$ as $x \to a$. Let x_n be any sequence such that (i) $x_n \to a$; (ii) $x_n \neq a$ for all n. Then $f(x_n) \to l$.

Rules 1.2-1.4 can be proved either directly using def 1.1 or by first applying 1.5 to reduce the situation to the corresponding theorem for sequences from M1P1.
Theorem 1.5 is useful in showing that limits do not exist. For example, if $f(x) = 1$ for rational x and $f(x) = 0$ otherwise, we can use Theorem 1.5 so show that f has no limit at any point. This theorem has the coverse:

(1.6) The following statements are equivalent:
(a) $f(x) \to l$ as $x \to a$.
(b) $f(x_n) \to l$ for every sequence x_n such that (i) $x_n \to a$, and (ii) $x_n \neq a$ for all n.

(1.7) If $f(x) \to l$, then $|f(x)| \to |l|$.

Rules 1.2-1.4 and 1.7 can be proved directly, or by using Theorem 1.6 and corresponding theorems for sequences from M1P1. These rules can be applied to continuous functions to see that if f and g are both continuous at a, then

(1.2') $f + g$ is continuous at a;
(1.3') fg is continuous at a;
(1.4') f/g is continuous at a provided $g(a) \neq 0$;
(1.7') $|f|$ is continuous at a.

(1.8) (*composition*) Suppose $g(x) \to l$ as $x \to a$ and f is continuous at a. Then $f(g(x)) \to f(l)$ as $x \to a$.
(1.8') (*continuity of compositions*) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. Here, as usual, $(f \circ g)(x) = f(g(x))$.

Remarks on Chapter 1:
(1) everything is valid with no change if x or $f(x)$ are complex.
(2) everything is valid for one-sided limits as well.

Chapter 2: Continuity on closed intervals.

When talking about intervals here, we always mean bounded intervals, i.e. boundaries a and b are finite real numbers.

Main definition: f is continuous on a closed interval $[a, b]$ if it is continuous at all points of (a, b), right continuous at a, and left continuous at b. For intervals $(a, b]$ or $[a, b)$ definitions are similar.

Basic observation: if f is continuous on a interval (α, β) which contains a subinterval $[a, b]$, then f is continuous on $[a, b]$.

Examples: polynomials are cts on any $[a, b]$, rational functions f/g are cts on any $[a, b]$ provided $g(x) \neq 0$ for all $x \in [a, b]$.
Theorem 1.5 has a counterpart for one-sided limits:
(2.0) If f is right continuous at a and x_n is any sequence such that $x_n \to a$ and $x_n \geq a$ for all n, then $f(x_n) \to f(a)$. (a similar statement holds for left continuous functions.)

Let f be continuous on a closed interval $[a, b]$. Then we have the following theorems:

(2.1) (*IVT: Intermediate value theorem*) f has the intermediate value property (IVP) on $[a, b]$, which means that if l is any number between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ such that $f(c) = l$.
Note that continuous functions on non-closed intervals do not have to have IVP and functions having IVP are not necessarily continuous.
(2.2) f is bounded (both above and below) on $[a, b]$. By definition, f is bounded above (below) on $[a, b]$, if there is a number M (m) such that $f(x) \leq M$ ($f(x) \geq m$), for all $x \in [a, b]$, resp.
(2.3) (*EVT: extreme value theorem*) There is some $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$. Also, there is some $c' \in [a, b]$ such that $f(x) \geq f(c')$ for all $x \in [a, b]$.

In these cases we say $f(c) = \max_{[a,b]} f, f(c') = \min_{[a,b]} f$, are maximum and minimum of f on $[a, b]$.

(2.4) *Inverse function theorem.* Let $f : [a, b] \to \mathbf{R}$ be strictly increasing and continuous on $[a, b]$. Let $[c, d] = f([a, b])$. Let's define the inverse function $g : [c, d] \to [a, b]$ by $x = g(y)$ iff $y = f(x)$. Then g is strictly increasing and continuous on $[c, d]$.

Chapter 2E: Exponential and log functions.

For any (real or complex) x we define $E(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$, the sum of an absolutely convergent series. It follows from this that $E(x+y) = E(x)E(y)$. Setting $e = E(1)$, it was shown in M1P1 that $E(x) = e^x$ for rational x. Here we have the following further properties of function E:

(2E1) $E(x) > 0$ for all $x \in \mathbf{R}$.
(2E2) $E(x)$ is strictly increasing on $\mathbf{R}$.
(2E3) $E(x)$ is continuous at every $x \in \mathbf{R}$.
(2E4) E takes every positive value exactly once.
By Theorem 2.4 we can define the inverse finction log by $x = \log y$ iff $E(x) = y$, for all positive $y > 0$.

(2E5) The function log is strictly increasing and continuous at all $y > 0$.

Using this, for any $a > 0$ we can define a^x by $a^x = E(x \log a)$. Then it is easy to see that $a^{x+y} = a^x a^y$, which implies that a^x agrees with the usual definition of a power for rational x (as in M1P1). From (2E3), (2E5) and Theorem 1.8', we get
(2E6) For any $a > 0$, the function a^x is continuous everywhere.

Chapter 3: Differentiation.

Suppose f is defined on some open interval containing a point a. We say f is differentiable at a if $\frac{f(a+h)-f(a)}{h}$ has a limit as $h \to 0$, and we call this limit $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. We say f is left (right) differentiable if the corresponding limits below exist and we set $f'_-(a) = \lim_{h\to 0-} \frac{f(a+h)-f(a)}{h}$ and $f'_+(a) = \lim_{h\to 0+} \frac{f(a+h)-f(a)}{h}$. The main fact relating differentiability and continuity is

(3.1) If f is differentiable at a, then f is continuous at a.

The converse is not true, as is shown by the function $f(x) = |x|$ at zero.

Rules for differentiation. Suppose f and g are both d'ble at a. Then

(3.2) $f + g$ is d'ble at a, and $(f + g)'(a) = f'(a) + g'(a)$.

(3.3) (*product rule*) fg is d'ble at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.

(3.4) if in addition $g(a) \neq 0$, then $1/g$ is d'ble at a, and $(1/g)'(a) = -g'(a)/g(a)^2$.

(3.5) (*quotient rule*) if in addition $g(a) \neq 0$, then f/g is d'ble at a, and $(f/g)'(a) = [f'(a)g(a) - f(a)g'(a)]/g(a)^2$.

(3.6) (*chain rule*) Suppose g is d'ble at a and f is d'ble at $g(a)$. Then $f \circ g$ is d'ble at a, and $(f \circ g)'(a) = f'(g(a))g'(a)$.

(3.7) (*inverse function*) Suppose $f : [a, b] \to [c, d]$ is continuous and strictly increasing, with inverse function $g : [c, d] \to [a, b]$. Suppose f is d'ble at some point $\alpha \in (a, b)$, and that $f'(\alpha) \neq 0$. Then g is d'ble at $\beta = f(\alpha)$, and $g'(\beta) = 1/f'(\alpha)$.

Chapter 4: Functions differentiable on an interval.

As before, we say that f has a (local) maximum at a, if there is a $\delta > 0$ such that $|x - a| < \delta$ implies $f(x) \leq f(a)$; and f has a (local) minimum at a, if there is a $\delta > 0$ such that $|x - a| < \delta$ implies $f(x) \geq f(a)$. In what follows, we will always mean local max and local min.

(4.1) Suppose f is d'ble at a, and that f has a (local) maximum or a minimum at a. Then $f'(a) = 0$.

For the following theorems, suppose f and g are continuous on a closed interval $[a, b]$, and differentiable on its interior (a, b). Then

(4.2) (*Rolle's Theorem*) Suppose in addition that $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.

(4.3) (*MVT: Mean value theorem*) There is a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(4.4) (*Cauchy's MVT*) There is a point $c \in (a, b)$ such that $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.

Applications of the MVT:

(4.5) Suppose $f'(x) = 0$ for all $x \in (a, b)$. Then f is constant on $[a, b]$.

(4.6) Suppose $f'(x) > 0$ for all $x \in (a, b)$. Then f is strictly increasing on $[a, b]$.

(4.7) (*Test for max/min*) Suppose that at some point c, $f'(c) = 0$ and $f''(c)$ exists. Then (i) if $f''(c) > 0$, f has a minimum at c; (ii) if $f''(c) < 0$, f has a maximum at c.

When calculating limits, it is often useful to use the following

(4.9) (*L'Hôpital's rule, one-sided version*) Suppose that

(i) f and g are both d'ble on $I = (a, a + \delta)$. (ii) $f(x) \to 0$ and $g(x) \to 0$ as $x \to a+$.

(iii) $g' \neq 0$ on I. (iv) $f'(x)/g'(x) \to l$ as $x \to a+$.

Then $f(x)/g(x) \to l$ as $x \to a+$.

Of course, there is a similar theorem for limits as $x \to a-$, and hence for two-sided limits as $x \to a$.

Taylor series: If f is a function such that $f^{(k)}(a)$ exist for all $k \in \mathbf{N}$, then $\sum_{k\geq 0} \frac{f^{(k)}(a)}{k!} h^k$ is called the Taylor series of f at a. Let P_{n-1} be the sum of the first n terms of the Taylor series, $P_{n-1}(h) = \sum_{k=0}^{n-1} \frac{f^{(k)}}{k!}(a)h^k$, and let $f(a+h) = P_{n-1}(h) + R_n(h)$. The we have the following estimates for the remainder R_n:

(4.9) (*Taylor's theorem with Lagrange's form of the remainder*) Fix $n \geq 1$ and let $h > 0$. Suppose that $f, f', \ldots, f^{(n-1)}$ are all continuous on $[a, a+h]$ and that $f^{(n)}$ exists on $(a, a+h)$. Then $R_n(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h)$, for some $\theta \in (0,1)$.
(4.9') (*Most useful form of Taylor's theorem*) Suppose f is n times d'ble on some open interval I containing a. Then if $a + h \in I$, we have $R_n(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h)$, for some $\theta \in (0,1)$.

Chapter 5: Riemann integration.

A partition Δ of $[a,b]$ is any finite subset of $[a,b]$ that contains a and b. If it has $n+1$ points, we write it in the form $\Delta = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$. Let f be a bounded function on $[a,b]$. Given a partition Δ, let $M_i = l.u.b._{x\in[x_{i-1},x_i]} f(x)$ and $m_i = g.l.b._{x\in[x_{i-1},x_i]} f(x)$. We define upper and lower Riemann sums by $S(f,\Delta) = \sum_{i=1}^n M_i(x_i - x_{i-1})$ and $s(f,\Delta) = \sum_{i=1}^n m_i(x_i - x_{i-1})$. Let us now define upper and lower Riemann integrals $J(f) = g.l.b.\{S(f,\Delta)$: over all finite partitions $\Delta\}$ and $j(f) = l.u.b.\{s(f,\Delta)$: over all finite partitions $\Delta\}$. Function f is Riemann integrable over $[a,b]$ if $j(f) = J(f)$. In that case we define $\int_a^b f = j(f) = J(f)$. We have the following properties:

(5.1) Let $M = l.u.b.\{f(x) : x \in [a,b]\}$, $m = g.l.b.\{f(x) : x \in [a,b]\}$. Then $m(b-a) \leq s(f,\Delta) \leq S(f,\Delta) \leq M(b-a)$.
(5.2) $j(f) \leq J(f)$ for any bounded f.
(5.3) If $\Delta' = \Delta \cup \{c\}$ is obtained from Δ by adding one more subdivision point c, then $S(f,\Delta') \leq S(f,\Delta)$ and $s(f,\Delta) \leq s(f,\Delta')$.
(5.4) Let Δ' be any subdivision of Δ (i.e. Δ' is obtained from Δ by adding more points, $\Delta \subset \Delta'$). Then $S(f,\Delta') \leq S(f,\Delta)$ and $s(f,\Delta) \leq s(f,\Delta')$.
(5.5) Let Δ_1 and Δ_2 be any two partitions of $[a,b]$. Then $s(f,\Delta_1) \leq S(f,\Delta_2)$.

The following are useful criteria for Riemann integrability:
(5.6) (*ϵ-criterion for Riemann integrability*) Suppose that for any $\epsilon > 0$ there is a partition Δ of $[a,b]$ such that $S(f,\Delta) - s(f,\Delta) < \epsilon$. Then f is Riemann integrable over $[a,b]$.
(5.7) Let f be monotonic on $[a,b]$. Then f is Riemann integrable over $[a,b]$.

To prove that all continuous functions are Riemann integrable, we need
(5.8) Let f be continuous on $[a,b]$. Then f is uniformly continuous on $[a,b]$, i.e. given any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in [a,b]$, condition $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
(5.9) Let f be continuous on $[a,b]$. Then f is Riemann integrable over $[a,b]$.

We have the following properties of the integral:
(5.10) (*linearity of the integral*) Let c be a constant and suppose f and g are both Riemann integrable over $[a,b]$. Then so is $cf + g$, and $\int_a^b (cf+g) = c\int_a^b f + \int_a^b g$.
(5.11) Let f and g be any bounded functions on $[a,b]$. Then $J(f+g) \leq J(f) + J(g)$ and $j(f+g) \geq j(f) + j(g)$.
(5.12) Let $a < c < b$ and suppose that f is continuous on $[a,b]$. Then $\int_a^b f = \int_a^c f + \int_c^b f$, where all integrals make sense.
(5.13) (*preparation for FTC*) Let f be continuous on $[a,b]$. Then f is Riemann integrable over $[a,x]$ for all $x \in [a,b]$, so it makes sense to define $\phi(x) = \int_a^x f$ (by convention, $\phi(a) = 0$). Then ϕ is d'ble on (a,b), continuous on $[a,b]$, and $\phi'(c) = f(c)$ for all $c \in (a,b)$.
(5.14) (*FTC: fundamental theorem of calculus*) Suppose f is continuous on $[a,b]$, and let F be any function that is continuous on $[a,b]$, d'ble on (a,b) and such that $F'(x) = f(x)$ for all $x \in (a,b)$. Then $\int_a^b f = F(b) - F(a)$.