# M2P1 Analysis II (2005) Dr M Ruzhansky

List of definitions, statements and examples.

## Chapter 1: Limits and continuity.

This chapter is mostly the revision of Chapter 6 of M1P1. First we consider functions $f(x)$ where $x$ and $f(x)$ are real.

(6.1)(M1P1) (*continuity*) Suppose $f$ is a function defined for all $x$ in some interval containing a point $a$. Then we say that $f$ *is continuous at* $a$ if given any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(6.2)(M1P1) (*limits*) Suppose $f$ is a function defined for all $x$ in some interval containing a point $a$, except (perhaps) at $a$ itself. Then "$f(x) \to l$ as $x \to a$" means that given any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$.

Here and in what follows $l$ should be a finite number. Note that the main and only difference between these two definitions is that when we take a limit of a function $f$ at a point $a$, we do not require $f(a)$ to be defined. That is why we have to require $0 < |x - a|$ to insure that $x \neq a$, and to replace $f(a)$ by a real number $l$ since $f(a)$ may be no longer defined. For the limit of $f$ at $a$, the values of $f$ around $a$, not at $a$, are important!
(6.3)(M1P1) Revise examples of continuous functions and limits.

Similarly, we can define *right and left limits*. So, "$f(x) \to l$ as $x \to a+$" means $\forall\epsilon > 0\ \exists\delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - l| < \epsilon$, and "$f(x) \to l$ as $x \to a-$" means $\forall\epsilon > 0$ $\exists\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - l| < \epsilon$, respectively.

Relation between limits and continuity: "$f$ is continuous (left cts, right cts) at $a$" means that (i) $f(a)$ is defined, and (ii) $f(x) \to f(a)$ as $x \to a$ ($x \to a-$, $x \to a+$, resp.)

Simple facts:
(i) $f(x) \to l$ as $x \to a$ if and only if $f(x) \to l$ as $x \to a-$ and $f(x) \to l$ as $x \to a+$.
(ii) $f$ is continuous at $a$ if and only if $f$ is left and right continuous at $a$.

(6.4)(M1P1) (*limits in terms of sequences*) The following statements are equivalent:
(a) $f(x) \to l$ as $x \to a$.
(b) $f(x_n) \to l$ for every sequence $x_n$ such that (i) $x_n \to a$, and (ii) $x_n \neq a$ for all $n$.

This theorem is useful in showing that limits do not exist. For example, if $f(x) = 1$ for rational $x$ and $f(x) = 0$ otherwise, we can use part (b) to show that $f$ has no limit at any point. Using the relation between limits and continuity, we get:

(6.5)(M1P1) (*continuity in terms of sequences*) The following statements are equivalent:
(a) $f$ is continuous at $a$.
(b) $f(x_n) \to f(a)$ for every sequence $x_n$ such that $x_n \to a$.

(6.6)(M1P1) (*basic rules for limits*) Suppose $f(x) \to l$ and $g(x) \to m$ as $x \to a$. Then
(i) $cf(x) \to cl$ for any $c \in \mathbb{R}$;
(ii) $f(x) + g(x) \to l + m$ as $x \to a$;
(iii) $f(x)g(x) \to lm$ as $x \to a$;
(iv) $f(x)/g(x) \to l/m$ as $x \to a$, provided $m \neq 0$;
(v) $|f(x)| \to |l|$ as $x \to a$.

This theorem can be proved either directly using $\epsilon$ and $\delta$, or by using characterisation (6.4)(M1P1) and corresponding theorems for sequences from M1P1. These rules can be applied to continuous functions:

(6.7)(M1P1) (*basic rules for continuity*) Suppose $f$ and $g$ are both continuous at $a$. Then
(i) $cf$ is continuous at $a$ for any $c \in \mathbb{R}$;
(ii) $f + g$ is continuous at $a$;
(iii) $fg$ is continuous at $a$;
(iv) $f/g$ is continuous at $a$, provided $g(a) \neq 0$;
(v) $|f|$ is continuous at $a$.

(6.8)(M1P1) (*limit of compositions*) Suppose $g(x) \to l$ as $x \to a$ and $f$ is continuous at $a$. Then $f(g(x)) \to f(l)$ as $x \to a$.
(6.9)(M1P1) (*continuity of compositions*) If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Here, as usual, $(f \circ g)(x) = f(g(x))$.

Remarks:
(1) everything is valid with no change if $x$ or $f(x)$ are complex.
(2) everything is valid for one-sided limits as well.

All of the theorems above can be also extended to higher dimensions. The only thing to change is the notion of the distance! Thus, for points $x, y \in \mathbb{R}^n$, we define

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. Then we can simply repeat definitions from M1P1 using this new distance where necessary.

(1.0) Let $x_n$ be a sequence of points in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We will say that $x_n \to x$ as $n \to \infty$ if $|x_n - x| \to 0$ as $n \to \infty$.
Here $|x_n - x|$ is a sequence of real numbers, so we already know from M1P1 how to understand its convergence to zero. We will also define a ball $B_\delta(a)$ with radius $\delta$ centred at $a$ by $B_\delta(a) = \{x \in \mathbb{R}^n : |x - a| < \delta\}$. Balls will now replace the intervals! We will consider real valued functions, i.e. $f(x)$ is real though $x$ is in $\mathbb{R}^n$. With this in mind, definitions (6.1) and (6.2) above become:

(1.1) (*continuity in* $\mathbb{R}^n$) Suppose $f$ is a function defined for all $x$ in some ball containing a point $a$. Then we say that *$f$ is continuous at $a$* if given any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

(1.2) (*limits in* $\mathbb{R}^n$) Suppose $f$ is a function defined for all $x$ in some ball containing a point $a$, except (perhaps) at $a$ itself. Then "$f(x) \to l$ as $x \to a$" means that given any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$.

(1.3) (*Main Theorem of Chapter 1*) Theorems (6.4)–(6.9) (M1P1) remain valid if we replace definitions (6.1) and (6.2)(M1P1) by definitions (1.1) and (1.2), respectively.

Remark: everything is still valid for functions $f : \mathbb{R}^n \to \mathbb{R}^n$ or $f : \mathbb{C}^n \to \mathbb{C}^n$ if we use the $n$-dimensional distance for values of $f$ as well.

## Chapter 2: Continuity on closed intervals.

In the next chapters we will work in one dimension and will assume that $x$ and $f(x)$ are real. When talking about intervals here, we always mean bounded intervals, i.e. boundaries $a$ and $b$ are finite real numbers, satisfying $-\infty < a \leq b < +\infty$. As usual, we define open and closed intervals by $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ and $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Main definition: $f$ is said to be *continuous on a closed interval* $[a, b]$ if it is continuous at all points of $(a, b)$, right continuous at $a$, and left continuous at $b$. For intervals $(a, b]$ or $[a, b)$ definitions are similar.

Trivial observation: if $f$ is continuous on an interval $(\alpha, \beta)$ which contains a subinterval $[a, b]$, then $f$ is continuous on $[a, b]$.

Examples: polynomials are continuous on any $[a, b]$, rational functions $f/g$ are continuous on any $[a, b]$ provided $g(x) \neq 0$ for all $x \in [a, b]$.

Theorems in Chapter 6 of M1P1 and in our Chapter 1 have counterparts for one-sided limits. For example, the counterpart of Theorem 6.4 (M1P1) is:
(2.0) Function $f$ is right continuous at $a$ if and only if $f(x_n) \to f(a)$ for every sequence $x_n$ such that (i) $x_n \to a$, and (ii) $x_n \geq a$ for all $n$. (a similar statement holds for left continuous functions)

We have the following important properties of continuous functions on closed intervals:

(2.1) (*IVT: Intermediate Value Theorem*) Let $f$ be continuous on a closed interval $[a, b]$. Then $f$ has the intermediate value property (IVP) on $[a, b]$, which means that if $l$ is any number between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ such that $f(c) = l$.
Note that continuous functions on non-closed intervals do not have to satisfy IVP and functions having IVP on closed intervals are not necessarily continuous. So these two notions are not at all equivalent.
(2.2) (*boundedness theorem*) Let $f$ be continuous on a closed interval $[a, b]$. Then $f$ *is bounded (both above and below) on* $[a, b]$, i.e. there is a number $M$ $(m)$ such that $f(x) \leq M$ $(f(x) \geq m)$, for all $x \in [a, b]$, resp.
(2.3) (*EVT: Extreme Value Theorem*) Let $f$ be continuous on a closed interval $[a, b]$. Then there is some $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$. Also, there is some $c' \in [a, b]$ such that $f(x) \geq f(c')$ for all $x \in [a, b]$.

In these cases we say $f(c) = \max_{[a,b]} f, f(c') = \min_{[a,b]} f$, are maximum and minimum of $f$ on $[a, b]$. Then we also say that $f$ attains its maximum and minimum at $c$ and $c'$, respectively.

(2.4) (*IFT: Inverse Function Theorem*) Let $f : [a, b] \to \mathbb{R}$ be strictly increasing and continuous on $[a, b]$. Let $[c, d] = f([a, b])$. Let us define the inverse function $g : [c, d] \to [a, b]$ by setting $g(y) = x$ if and only if $f(x) = y$. Then $g$ is strictly increasing and continuous on $[c, d]$.

Exponential and log functions.
For any (real or complex) $x$ we define $E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, the sum of an absolutely convergent series. It follows from this definition that $E(x+y) = E(x)E(y)$. Setting $e = E(1)$, it was shown in M1P1 that $E(x) = e^x$ for rational $x$. For irrational $x$, we can define $e^x$ to be equal to $E(x)$.

(2.5) We have the following further properties of the exponential function $E(x)$:
(i) $E(x) > 0$ for all $x \in \mathbb{R}$;
(ii) $E(x)$ is strictly increasing on $\mathbb{R}$;
(iii) $E(x)$ is continuous at every $x \in \mathbb{R}$;
(iv) $E$ takes every positive value exactly once.
By IFT (2.4) we can define the inverse function log by setting $x = \log y$ if and only if $E(x) = y$, for all positive $y > 0$, and obtain

(2.6) The function log is strictly increasing and continuous at all $y > 0$.

Using this, for any $a > 0$ we can define $a^x$ by $a^x = E(x \log a)$. Then it is easy to see that $a^{x+y} = a^x a^y$, which implies that $a^x$ agrees with the usual definition of a power for rational $x$ (as in M1P1). From (2.5) and (6.9)(M1P1) we immediately get
(2.7) For any $a > 0$, the function $a^x$ is continuous everywhere.

## Chapter 3: Differentiation.

Suppose $f$ is defined on some open interval containing a point $a$. We say $f$ is *differentiable at* $a$ if $\frac{f(a+h)-f(a)}{h}$ has a limit as $h \to 0$, and we call this limit $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$. We say $f$ is *left (right) differentiable at* $a$ if the corresponding limits below exist and we set $f'_-(a) = \lim_{h\to 0-} \frac{f(a+h)-f(a)}{h}$ and $f'_+(a) = \lim_{h\to 0+} \frac{f(a+h)-f(a)}{h}$.
We can note that although expressions under the limit signs are undefined for $h = 0$ (they are of the form $0/0$), we can still talk about limits of these expressions as $h \to 0$, since we do not have to look at what happens at $h = 0$ according to definition of limits (6.2)(M1P1).

Simple observation: Function $f$ is differentiable at $a$ if and only if it is both left and right differentiable at $a$ and $f'_+(a) = f'_-(a)$. In this case we also have $f'(a) = f'_+(a) = f'_-(a)$.
The main fact relating differentiability and continuity is

(3.1) If $f$ is differentiable at $a$, then $f$ is continuous at $a$.
The converse is not true, as is shown by the function $f(x) = |x|$ at zero.

Rules for differentiation. Suppose $f$ and $g$ are both differentiable at $a$. Then
(3.2) $f + g$ is d'ble at $a$, and $(f + g)'(a) = f'(a) + g'(a)$.
(3.3) (*product rule*) $fg$ is d'ble at $a$, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.
(3.4) if in addition $g(a) \neq 0$, then $1/g$ is d'ble at $a$, and $(1/g)'(a) = -g'(a)/g(a)^2$.
(3.5) (*quotient rule*) if in addition $g(a) \neq 0$, then $f/g$ is d'ble at $a$, and $(f/g)'(a) = [f'(a)g(a) - f(a)g'(a)]/g(a)^2$.
(3.6) (*chain rule*) Suppose $g$ is d'ble at $a$ and $f$ is d'ble at $g(a)$. Then $f \circ g$ is d'ble at $a$, and $(f \circ g)'(a) = f'(g(a))g'(a)$.
(3.7) (*inverse function*) Suppose $f : [a, b] \to [c, d]$ is continuous and strictly increasing, with inverse function $g : [c, d] \to [a, b]$. Suppose $f$ is d'ble at some point $\alpha \in (a, b)$, and that $f'(\alpha) \neq 0$. Then $g$ is d'ble at $\beta = f(\alpha)$, and $g'(\beta) = 1/f'(\alpha)$.
In fact, the formula for the derivative of the inverse function simply follows from its differentiability and the chain rule. Indeed, differentiating the equality $g(f(x)) = x$ at $\alpha$, by the chain rule we get $g'(f(\alpha))f'(\alpha) = 1$, which implies $g'(\beta) = 1/f'(\alpha)$ in (3.7). This argument also shows that condition $f'(\alpha) \neq 0$ is necessary for the differentiability of $g$ at $f(a)$.

## Chapter 4: Functions differentiable on an interval.

As before, we say that $f$ has a *(local) maximum* at $a$, if there is a $\delta > 0$ such that $|x - a| < \delta$ implies $f(x) \leq f(a)$; and $f$ has a *(local) minimum* at $a$, if there is a $\delta > 0$ such that $|x - a| < \delta$ implies $f(x) \geq f(a)$. In what follows, we will always mean local maximum and local minimum.

(4.1) (*derivative at local extrema is zero*) Suppose $f$ is differentiable at $a$, and that $f$ has a (local) maximum or a minimum at $a$. Then $f'(a) = 0$.

For the following theorems (4.2)-(4.4), suppose $f$ and $g$ are continuous on a closed interval $[a, b]$, and differentiable on its interior $(a, b)$. Then
(4.2) (*Rolle's Theorem*) Suppose in addition that $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.
(4.3) (*MVT: Mean Value Theorem*) There is a point $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
(4.4) (*Cauchy's MVT*) There is a point $c \in (a, b)$ such that $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.

Applications of the MVT:
(4.5) Suppose $f'(x) = 0$ for all $x \in (a, b)$. Then $f$ is constant on $[a, b]$.

(4.6) Suppose $f'(x) > 0$ for all $x \in (a, b)$. Then $f$ is strictly increasing on $[a, b]$.
(4.7) (*Test for max/min*) Suppose that at some point $c$, $f'(c) = 0$ and $f''(c)$ exists. Then (i) if $f''(c) > 0$, $f$ has a minimum at $c$; (ii) if $f''(c) < 0$, $f$ has a maximum at $c$.

When calculating limits, it is often useful to use the following
(4.8) (*L'Hôpital's rule, one-sided version*) Suppose that
(i) $f$ and $g$ are both d'ble on $I = (a, a + \delta)$;
(ii) $f(x) \to 0$ and $g(x) \to 0$ as $x \to a+$;
(iii) $g' \neq 0$ on $I$;
(iv) $f'(x)/g'(x) \to l$ as $x \to a+$.
Then $f(x)/g(x) \to l$ as $x \to a+$.

Of course, there is a similar theorem for limits as $x \to a-$, and hence also for two-sided limits as $x \to a$.

Taylor series: By $f^{(k)}(a)$ we will denote the $k$-th order derivative of $f$ at $a$. If $f$ is a function such that $f^{(k)}(a)$ exist for all $k \in \mathbf{N}$, then $\sum_{k\geq 0} \frac{f^{(k)}(a)}{k!} h^k$ is called the *Taylor series of $f$ at $a$*. Let $P_{n-1}$ be the sum of the first $n$ terms of the Taylor series, i.e. $P_{n-1}(h) = \sum_{k=0}^{n-1} \frac{f^{(k)}}{k!}(a)h^k$, and let us define $R_n(h)$ by the equality $f(a+h) = P_{n-1}(h)+R_n(h)$. The we have the following estimates for the remainder $R_n$:

(4.9) (*Taylor's theorem with Lagrange's form of the remainder*) Fix $n \geq 1$ and let $h > 0$. Suppose that $f, f', \ldots, f^{(n-1)}$ are all continuous on $[a, a+h]$ and that $f^{(n)}$ exists on $(a, a+h)$. Then $R_n(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h)$, for some $\theta \in (0, 1)$.
(4.10) (*Most useful form of Taylor's theorem*) Suppose $f$ is $n$ times d'ble on some open interval $I$ containing $a$. Then if $a + h \in I$, we have $R_n(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h)$, for some $\theta \in (0, 1)$.

## Chapter 5: Riemann integration.

A *partition* $\Delta$ of $[a, b]$ is (by definition) any finite subset of $[a, b]$ that contains $a$ and $b$. If it has $n + 1$ points, we write it in the form $\Delta = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$. Let $f$ be a bounded function on $[a, b]$. Given a partition $\Delta$, let $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$ and $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$. We define *upper and lower Riemann sums* by $S(f, \Delta) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ and $s(f, \Delta) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$. Let us now define *upper and lower Riemann integrals* $J(f) = \inf\{S(f, \Delta) :$ over all finite partitions $\Delta\}$ and $j(f) = \sup\{s(f, \Delta) :$ over all finite partitions $\Delta\}$. Function $f$ is called *Riemann integrable over* $[a, b]$ if $j(f) = J(f)$. In that case we define its *Riemann integral* by $\int_a^b f = j(f) = J(f)$. We have the following properties:

(5.1) Let $M = \sup\{f(x) : x \in [a, b]\}$, $m = \inf\{f(x) : x \in [a, b]\}$. Then $m(b - a) \leq s(f, \Delta) \leq S(f, \Delta) \leq M(b - a)$.
(5.2) $j(f) \leq J(f)$ for any bounded function $f$ on $[a, b]$.
(5.3) If $\Delta' = \Delta \cup \{c\}$ is obtained from $\Delta$ by adding one more subdivision point $c$, then $S(f, \Delta') \leq S(f, \Delta)$ and $s(f, \Delta) \leq s(f, \Delta')$.
(5.4) Let $\Delta'$ be any subdivision of $\Delta$ (i.e. $\Delta'$ is obtained from $\Delta$ by adding more points, which also means $\Delta \subset \Delta'$). Then $S(f, \Delta') \leq S(f, \Delta)$ and $s(f, \Delta) \leq s(f, \Delta')$.
(5.5) Let $\Delta_1$ and $\Delta_2$ be any two partitions of $[a, b]$. Then $s(f, \Delta_1) \leq S(f, \Delta_2)$.

The following are useful criteria for Riemann integrability:
(5.6) (*$\epsilon$-criterion for Riemann integrability*) Suppose that for any $\epsilon > 0$ there is a partition $\Delta$ of $[a, b]$ such that $S(f, \Delta) - s(f, \Delta) < \epsilon$. Then $f$ is Riemann integrable over $[a, b]$.
(5.7) Let $f$ be monotonic on $[a, b]$. Then $f$ is Riemann integrable over $[a, b]$.

To prove that all continuous functions are Riemann integrable, we need
(5.8) Let $f$ be continuous on $[a, b]$. Then $f$ is *uniformly continuous on* $[a, b]$, i.e. given any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in [a, b]$, condition $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
(5.9) Let $f$ be continuous on $[a, b]$. Then $f$ is Riemann integrable over $[a, b]$.

We have the following further properties of the Riemann integral:
(5.10) (*linearity of the integral*) Let $\alpha, \beta \in \mathbb{R}$ and suppose $f$ and $g$ are both Riemann integrable over $[a, b]$. Then so is $\alpha f + \beta g$, and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.
(5.11) Let $f$ and $g$ be any bounded functions on $[a, b]$. Then $J(f + g) \leq J(f) + J(g)$ and $j(f + g) \geq j(f) + j(g)$.
(5.12) Let $a < c < b$ and suppose that $f$ is continuous on $[a, b]$. Then $\int_a^b f = \int_a^c f + \int_c^b f$, where all integrals make sense.
(5.13) (*preparation for FTC*) Let $f$ be continuous on $[a, b]$. Then $f$ is Riemann integrable over $[a, x]$ for all $x \in [a, b]$, so it makes sense to define $\phi(x) = \int_a^x f$ (by convention, $\phi(a) = 0$). Then $\phi$ is d'ble on $(a, b)$, continuous on $[a, b]$, and $\phi'(c) = f(c)$ for all $c \in (a, b)$.
(5.14) (*FTC: Fundamental Theorem of Calculus*) Suppose $f$ is continuous on $[a, b]$, and let $F$ be any function that is continuous on $[a, b]$, differentiable on $(a, b)$ and such that $F'(x) = f(x)$ for all $x \in (a, b)$. Then $\int_a^b f = F(b) - F(a)$.