M2PM1 Analysis II (2008) Dr M Ruzhansky
List of definitions, statements and examples
Preliminary version

Chapter 0: Some revision of M1P1: Limits and continuity.

This chapter is mostly the revision of Chapter 6 of M1P1. First we consider functions $f : \mathbb{R}^n \to \mathbb{R}$ where $x \in \mathbb{R}^n$ and where $f(x) \in \mathbb{R}$ is real.

For points $x, y \in \mathbb{R}^n$, we define the (Euclidean) distance

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$. First we observe that we can define convergence of points in $\mathbb{R}^n$ using the notion of convergence of real numbers.

(0.1) Let x_k be a sequence of points in $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We will say that $x_k \to x$ as $k \to \infty$ if $|x_k - x| \to 0$ as $k \to \infty$. This means that for every $\epsilon > 0$ there is some K such that for all $k \geq K$ we have $|x_k - x| < \epsilon$. We also define a ball $B_\delta(a)$ with radius δ centred at a by $B_\delta(a) = \{x \in \mathbb{R}^n : |x - a| < \delta\}$. Balls will now replace the intervals in $\mathbb{R}$. We will consider real valued functions, i.e. $f(x)$ is real though x is in $\mathbb{R}^n$.

(0.2) (*continuity in* $\mathbb{R}^n$) Suppose f is a function defined for all x in some ball containing a point a. Then we say that f *is continuous at* a if given any $\epsilon > 0$, there is a $\delta > 0$ such that $|x-a| < \delta$ implies $|f(x)-f(a)| < \epsilon$.

(0.3) (*limits in* $\mathbb{R}^n$) Suppose f is a function defined for all x in some ball containing a point a, except (perhaps) at a itself. Then "$f(x) \to l$ as $x \to a$" means that given any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - l| < \epsilon$.

Here and in what follows l should be a finite number. Note that the main and only difference between these two definitions is that when we take a limit of a function f at a point a, we do not require $f(a)$ to be defined. That is why we have to require $0 < |x-a|$ to insure that $x \neq a$, and to replace $f(a)$ by a real number l since $f(a)$ may be no longer defined. For the limit of f at a, the values of f around a, not at a, are important!

Relation between limits and continuity: "f is continuous at a" means that (i) $f(a)$ is defined, and (ii) $f(x) \to f(a)$ as $x \to a$.

(0.4) (*limits in terms of sequences*) The following statements are equivalent:
(a) $f(x) \to l$ as $x \to a$.
(b) $f(x_k) \to l$ for every sequence x_k such that (i) $x_k \to a$, and (ii) $x_k \neq a$ for all k.

This theorem is useful in showing that limits do not exist. For example, in $\mathbb{R}$, if $f(x) = 1$ for rational x and $f(x) = 0$ otherwise, we can use part (b) to show that f has no limit at any point. Using the relation between limits and continuity, we get:

(0.5) (*continuity in terms of sequences*) The following statements are equivalent:
(a) f is continuous at a.
(b) $f(x_k) \to f(a)$ for every sequence x_k such that $x_k \to a$.

(0.6) (*basic rules for limits*) Suppose $f(x) \to l$ and $g(x) \to m$ as $x \to a$. Then
(i) $cf(x) \to cl$ for any $c \in \mathbb{R}$;
(ii) $f(x) + g(x) \to l + m$ as $x \to a$;
(iii) $f(x)g(x) \to lm$ as $x \to a$;
(iv) $f(x)/g(x) \to l/m$ as $x \to a$, provided $m \neq 0$;
(v) $|f(x)| \to |l|$ as $x \to a$.

This theorem can be proved either directly using ϵ and δ, or by using characterisation (0.4) and corresponding theorems for sequences from M1P1. These rules can be applied to continuous functions:

(0.7) (*basic rules for continuity*) Suppose f and g are both continuous at a. Then
(i) cf is continuous at a for any $c \in \mathbb{R}$;
(ii) $f+g$ is continuous at a;
(iii) fg is continuous at a;
(iv) f/g is continuous at a, provided $g(a) \neq 0$;
(v) $|f|$ is continuous at a.

(0.8) (*limit of compositions*) Suppose $g(x) \to l$ as $x \to a$ and f is continuous at l. Then $f(g(x)) \to f(l)$ as $x \to a$.
(0.9) (*continuity of compositions*) If g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a. Here, as usual, $(f \circ g)(x) = f(g(x))$.

Remark: everything is valid with no change for functions $f : \mathbb{R}^n \to \mathbb{R}^m$ or $f : \mathbb{C}^n \to \mathbb{C}^m$ if we use the m-dimensional distance for values of f as well: the same definition as in $\mathbb{R}^n$ also for the distance in $\mathbb{C}^n$.

Continuity on closed intervals in $\mathbb{R}$.

From now on until the last chapter, we will work in one dimension and will assume that x and $f(x)$ are real. *Thus, we set the dimension* $n = 1$.

Similarly to above, we can define *right and left limits*. So, the limit from the right "$f(x) \to l$ as $x \to a+$" means $\forall \epsilon > 0\ \exists \delta > 0$ such that $a < x < a + \delta$ implies $|f(x) - l| < \epsilon$, and the limit from the left "$f(x) \to l$ as $x \to a-$" means $\forall \epsilon > 0\ \exists \delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - l| < \epsilon$, respectively.

Relation between limits and continuity: "f is left continuous (right continuous) at a" means that (i) $f(a)$ is defined, and (ii) $f(x) \to f(a)$ as $x \to a-$ ($x \to a+$, respectively)

Simple facts:
(i) $f(x) \to l$ as $x \to a$ if and only if $f(x) \to l$ as $x \to a-$ and $f(x) \to l$ as $x \to a+$.
(ii) f is continuous at a if and only if f is left and right continuous at a.

In $\mathbb{R}$, when talking about intervals, we always mean bounded intervals, i.e. boundaries a and b are finite real numbers, satisfying $-\infty < a \leq b < +\infty$. As usual, we define open and closed intervals by $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ and $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

(0.10) Main definition: f is said to be *continuous on a closed interval* $[a, b]$ if it is continuous at all points of (a, b), right continuous at a, and left continuous at b. For intervals $(a, b]$ or $[a, b)$ definitions are similar.

Trivial observation: if f is continuous on an interval (α, β) which contains a subinterval $[a, b]$, then f is continuous on $[a, b]$.

Examples: polynomials are continuous on any $[a, b]$, rational functions f/g are continuous on any $[a, b]$ provided $g(x) \neq 0$ for all $x \in [a, b]$.

Theorems above have counterparts for one-sided limits. For example, the counterpart of Theorem 0.4 is:

(0.11) Function f is right continuous at a if and only if $f(x_k) \to f(a)$ for every sequence x_k such that (i) $x_k \to a$, and (ii) $x_n \geq a$ for all k. (a similar statement holds for left continuous functions)

We have the following important properties of continuous functions on closed intervals:

(0.11) (*IVT: Intermediate Value Theorem*) Let f be continuous on a closed interval $[a, b]$. Then f has the intermediate value property (IVP) on $[a, b]$, which means that if l is any number between $f(a)$ and $f(b)$, then there is some $c \in [a, b]$ such that $f(c) = l$.
Note that continuous functions on non-closed intervals do not have to satisfy IVP and functions having IVP on closed intervals are not necessarily continuous. So these two notions are not at all equivalent.

(0.12) (*boundedness theorem*) Let f be continuous on a closed interval $[a,b]$. Then f *is bounded (both above and below) on* $[a,b]$, i.e. there is a number M (m) such that $f(x) \leq M$ $(f(x) \geq m)$, for all $x \in [a,b]$, resp.
(0.13) (*EVT: Extreme Value Theorem*) Let f be continuous on a closed interval $[a,b]$. Then there is some $c \in [a,b]$ such that $f(x) \leq f(c)$ for all $x \in [a,b]$. Also, there is some $c' \in [a,b]$ such that $f(x) \geq f(c')$ for all $x \in [a,b]$.

In these cases we write $f(c) = \max_{[a,b]} f, f(c') = \min_{[a,b]} f$, are maximum and minimum of f on $[a,b]$. Then we also say that f attains its maximum and minimum at c and at c', respectively.

(0.14) (*IFT: Inverse Function Theorem*) Let $f : [a,b] \to \mathbb{R}$ be strictly increasing and continuous on $[a,b]$. Let $[c,d] = f([a,b])$. Let us define the inverse function $g : [c,d] \to [a,b]$ by setting $g(y) = x$ if and only if $f(x) = y$. Then g is strictly increasing and continuous on $[c,d]$.

Summary of exponential and log functions given here for convenience (Revise M1P1 for full details)
For any (real or complex) x we define $E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, the sum of an absolutely convergent series. It follows from this definition that $E(x+y) = E(x)E(y)$. Setting $e = E(1)$, it was shown in M1P1 that $E(x) = e^x$ for rational x. For irrational x, we can define e^x to be equal to $E(x)$.

(0.15) We have the following further properties of the exponential function $E(x)$:
(i) $E(x) > 0$ for all $x \in \mathbb{R}$;
(ii) $E(x)$ is strictly increasing on $\mathbb{R}$;
(iii) $E(x)$ is continuous at every $x \in \mathbb{R}$;
(iv) E takes every positive value exactly once.
These properties have been established in M1P1. For example, property (iv) follows from properties (ii) and (iii) by the intermediate value theorem, etc.

By IFT (0.14) we can define the inverse function log by setting $x = \log y$ if and only if $E(x) = y$, for all positive $y > 0$, and obtain

(0.16) The function log is strictly increasing and continuous at all $y > 0$.

Using this, for any $a > 0$ we can define a^x by $a^x = E(x \log a)$. Then it is easy to see that $a^{x+y} = a^x a^y$, which implies that a^x agrees with the usual definition of a power for rational x (as in M1P1). From (0.15) and (0.9) we immediately get
(0.17) For any $a > 0$, the function a^x is continuous everywhere.

Chapter 1: Differentiation.

Suppose f is defined on some open interval containing a point a. We say that f is *differentiable at* a if $\frac{f(a+h)-f(a)}{h}$ has a limit as $h \to 0$, and we call this limit $f'(a)$, so that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

We say f is *left (right) differentiable at* a if the corresponding limits below exist and we set $f'_-(a) = \lim_{h \to 0-} \frac{f(a+h)-f(a)}{h}$ and $f'_+(a) = \lim_{h \to 0+} \frac{f(a+h)-f(a)}{h}$.
We can note that although expressions under the limit signs are undefined for $h = 0$ (they are of the form $0/0$), we can still talk about limits of these expressions as $h \to 0$, since we do not have to look at what happens at $h = 0$ according to definition of limits (0.3).

Simple observation: Function f is differentiable at a if and only if it is both left and right differentiable at a and $f'_+(a) = f'_-(a)$. In this case we also have $f'(a) = f'_+(a) = f'_-(a)$.
The main fact relating differentiability and continuity is

(1.1) If f is differentiable at a, then f is continuous at a.
The converse is not true, as is shown e.g. by the function $f(x) = |x|$ at zero.

Rules for differentiation. Suppose f and g are both differentiable at a. Then

(1.2) $f+g$ is differentiable at a, and $(f+g)'(a) = f'(a) + g'(a)$.
(1.3) (*product rule*) fg is differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.
(1.4) if in addition $g(a) \neq 0$, then $1/g$ is differentiable at a, and $(1/g)'(a) = -g'(a)/g(a)^2$.
(1.5) (*quotient rule*) if in addition $g(a) \neq 0$, then f/g is differentiable at a, and

$$(f/g)'(a) = [f'(a)g(a) - f(a)g'(a)]/g(a)^2.$$

(1.6) (*chain rule*) Suppose g is differentiable at a and f is differentiable at $g(a)$. Then $f \circ g$ is differentiable at a, and $(f \circ g)'(a) = f'(g(a))g'(a)$.
(1.7) (*inverse function*) Suppose $f : [a,b] \to [c,d]$ is continuous and strictly increasing, with inverse function $g : [c,d] \to [a,b]$. Suppose f is differentiable at some point $\alpha \in (a,b)$, and that $f'(\alpha) \neq 0$. Then g is differentiable at $\beta = f(\alpha)$, and $g'(\beta) = 1/f'(\alpha)$.
In fact, the formula for the derivative of the inverse function simply follows from its differentiability and the chain rule. Indeed, differentiating the equality $g(f(x)) = x$ at α, by the chain rule we get $g'(f(\alpha))f'(\alpha) = 1$, which implies $g'(\beta) = 1/f'(\alpha)$ in (1.7). This argument also shows that condition $f'(\alpha) \neq 0$ is necessary for the differentiability of g at $f(a)$. Note that in (1.7) we still have to prove that g is differentiable at β.

Chapter 2: Functions differentiable on an interval.

As before, we say that f has a *(local) maximum* at a, if there is a $\delta > 0$ such that $|x-a| < \delta$ implies $f(x) \leq f(a)$; and f has a *(local) minimum* at a, if there is a $\delta > 0$ such that $|x-a| < \delta$ implies $f(x) \geq f(a)$. In what follows, we will always mean local maximum and local minimum.

(2.1) (*derivative at a local extrema is zero*) Suppose f is differentiable at a, and that f has a (local) maximum or a (local) minimum at a. Then $f'(a) = 0$.

For the following theorems (2.2)-(2.4), suppose f and g are continuous on a closed interval $[a,b]$, and differentiable on its interior (a,b). Then
(2.2) (*Rolle's Theorem*) Suppose in addition that $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a,b)$.
(2.3) (*MVT: Mean Value Theorem*) There is a point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.
(2.4) (*Cauchy's MVT*) There is a point $c \in (a,b)$ such that $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.

Applications of the MVT:
(2.5) Suppose $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant on $[a,b]$.
(2.6) Suppose $f'(x) > 0$ for all $x \in (a,b)$. Then f is strictly increasing on $[a,b]$.
(2.7) (*Test for max/min*) Suppose that at some point c, $f'(c) = 0$ and $f''(c)$ exists. Then (i) if $f''(c) > 0$, f has a minimum at c; (ii) if $f''(c) < 0$, f has a maximum at c.

When calculating limits, it is often very helpful to use the following
(2.8) (*L'Hôpital's rule, one-sided version*) Suppose that
(i) f and g are both differentiable on $I = (a, a+\delta)$;
(ii) $f(x) \to 0$ and $g(x) \to 0$ as $x \to a+$;
(iii) $g' \neq 0$ on I;
(iv) $f'(x)/g'(x) \to l$ as $x \to a+$.
Then $f(x)/g(x) \to l$ as $x \to a+$.

Of course, there is a similar theorem for limits as $x \to a-$, and hence also for two-sided limits as $x \to a$.

Taylor series: By $f^{(k)}(a)$ we will denote the k-th order derivative of f at a. If f is a function such that $f^{(k)}(a)$ exist for all $k \in \mathbb{N}$, then $\sum_{k\geq 0} \frac{f^{(k)}(a)}{k!} h^k$ is called the *Taylor series of f at a*. Let P_{n-1} be the sum of the first n terms of the Taylor series, i.e.

$$P_{n-1}(h) = \sum_{k=0}^{n-1} \frac{f^{(k)}}{k!}(a)h^k$$

is the polynomial of order $n-1$ in h, and let us define $R_n(h)$ by the equality $f(a+h) = P_{n-1}(h) + R_n(h)$. Then we have the following estimates for the remainder R_n:

(2.9) (*Taylor's theorem with Lagrange's form of the remainder*) Fix $n \geq 1$ and let $h > 0$. Suppose that $f, f', \ldots, f^{(n-1)}$ are all continuous on $[a, a+h]$ and that $f^{(n)}$ exists on $(a, a+h)$. Then

$$R_n(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h),$$

for some $\theta \in (0, 1)$.
(2.10) (*Most useful form of Taylor's theorem*) Suppose f is n times differentiable on some open interval I containing a. Then if $a + h \in I$, we have $R_n(h) = \frac{h^n}{n!} f^{(n)}(a + \theta h)$, for some $\theta \in (0, 1)$.

Chapter 3: Riemann integration.

A *partition* Δ of $[a, b]$ is (by definition) any finite subset of $[a, b]$ that contains a and b. If it has $n+1$ points, we write it in the form $\Delta = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$. Let f be a bounded function on $[a, b]$. Given a partition Δ, let $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ and $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$. We define *upper and lower Riemann sums* by $S(f, \Delta) = \sum_{i=1}^n M_i(x_i - x_{i-1})$ and $s(f, \Delta) = \sum_{i=1}^n m_i(x_i - x_{i-1})$. Let us now define *upper and lower Riemann integrals* $J(f) = \inf\{S(f, \Delta) : \text{over all finite partitions } \Delta\}$ and $j(f) = \sup\{s(f, \Delta) : \text{over all finite partitions } \Delta\}$. Function f is called *Riemann integrable over* $[a, b]$ if $j(f) = J(f)$. In that case we define its *Riemann integral* by $\int_a^b f = j(f) = J(f)$. We have the following properties:

(3.1) Let $M = \sup\{f(x) : x \in [a, b]\}$, $m = \inf\{f(x) : x \in [a, b]\}$. Then $m(b-a) \leq s(f, \Delta) \leq S(f, \Delta) \leq M(b-a)$.
(3.2) $j(f) \leq J(f)$ for any bounded function f on $[a, b]$.
(3.3) If $\Delta' = \Delta \cup \{c\}$ is obtained from Δ by adding one more subdivision point c, then $S(f, \Delta') \leq S(f, \Delta)$ and $s(f, \Delta) \leq s(f, \Delta')$.
(3.4) Let Δ' be any subdivision of Δ (i.e. Δ' is obtained from Δ by adding more points, which also means $\Delta \subset \Delta'$). Then $S(f, \Delta') \leq S(f, \Delta)$ and $s(f, \Delta) \leq s(f, \Delta')$.
(3.5) Let Δ_1 and Δ_2 be any two partitions of $[a, b]$. Then $s(f, \Delta_1) \leq S(f, \Delta_2)$.

The following are useful criteria for Riemann integrability:
(3.6) (*ϵ-criterion for Riemann integrability*) Suppose that for any $\epsilon > 0$ there is a partition Δ of $[a, b]$ such that $S(f, \Delta) - s(f, \Delta) < \epsilon$. Then f is Riemann integrable over $[a, b]$.
(3.7) Let f be monotonic on $[a, b]$. Then f is Riemann integrable over $[a, b]$.

To prove that all continuous functions are Riemann integrable, we need
(3.8) Let f be continuous on $[a, b]$. Then f is *uniformly continuous on* $[a, b]$, i.e. given any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x, y \in [a, b]$, condition $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.
(3.9) Let f be continuous on $[a, b]$. Then f is Riemann integrable over $[a, b]$.

We have the following further properties of the Riemann integral:
(3.10) (*linearity of the integral*) Let $\alpha, \beta \in \mathbb{R}$ and suppose f and g are both Riemann integrable over $[a, b]$. Then so is $\alpha f + \beta g$, and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.
(3.11) Let f and g be any bounded functions on $[a, b]$. Then $J(f+g) \leq J(f) + J(g)$ and $j(f+g) \geq j(f) + j(g)$.
(3.12) Let $a < c < b$ and suppose that f is continuous on $[a, b]$. Then $\int_a^b f = \int_a^c f + \int_c^b f$, where all integrals make sense.
(3.13) (*preparation for FTC*) Let f be continuous on $[a, b]$. Then f is Riemann integrable over $[a, x]$ for all $x \in [a, b]$, so it makes sense to define $\phi(x) = \int_a^x f$ (by convention, $\phi(a) = 0$). Then ϕ is differentiable on (a, b), continuous on $[a, b]$, and $\phi'(c) = f(c)$ for all $c \in (a, b)$.
(3.14) (*FTC: Fundamental Theorem of Calculus*) Suppose f is continuous on $[a, b]$, and let F be any function that is continuous on $[a, b]$, differentiable on (a, b) and such that $F'(x) = f(x)$ for all $x \in (a, b)$. Then $\int_a^b f = F(b) - F(a)$.

Here we will view $\mathbb{R}^n$ as a vector space equipped with the inner product $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ and the length $|x| = \sqrt{x \cdot x}$. Recall the Cauchy-Schwartz inequality $|x \cdot y| \leq |x||y|$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Revise continuity and limits for functions in $\mathbb{R}^n$ from M1P1 (see also Chapter 0).

(4.1) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous at $a \in \mathbb{R}^n$. Let $a = (a_1, \ldots, a_n)$. Then the function $x_k \mapsto f(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_n)$ is continuous at $a_k \in \mathbb{R}$ in the usual way.

The converse to (4.1) need not be true. Let us define a function g on $\mathbb{R}^2$ by $g(x, y) = \frac{xy}{x^2+y^2}$ for $(x, y) \neq (0, 0)$ and by $g(0, 0) = 0$. Then the following is true for this g:
(4.2) Function g is continuous at any point except $(0, 0)$. It is not continuous at $(0, 0)$.
(4.3) For any fixed $x \in \mathbb{R}$, the function $y \mapsto g(x, y)$ is continuous on $\mathbb{R}$, and for any fixed $y \in \mathbb{R}$, the function $x \mapsto g(x, y)$ is continuous on $\mathbb{R}$.
Conclusion: a function g may be continuous separately in x and y, without being continuous in (x, y).

(4.4) A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *differentiable at a point* $a \in \mathbb{R}^n$ if there exists $v \in \mathbb{R}^n$ such that $\boxed{f(a + h) - f(a) = v \cdot h + o(h)}$ where $o(h)$ denotes any function such that $\frac{o(h)}{|h|} \to 0$ as $h \to 0$. In other words, if $[f(a + h) - f(a) - v \cdot h]/|h| \to 0$ as $h \to 0$. The vector v is called *the gradient* of f at a, and it is denoted by $\nabla f(a)$ or by $\frac{df}{dx}(a)$.
Note that in the case $n = 1$ we could simply divide this definition by h to recover the definition in Chapter 1. Here we can not simply divide by h because it is a vector.

Partial derivatives. By fixing all coordinates but x_k, one can consider a function $x_k \mapsto f(a_1, \ldots, a_{k-1}, x_k, a_{k+1}, \ldots, a_n)$, as in (4.1). This is a function on $\mathbb{R}$ and can be differentiated in the usual way. Such derivative is denoted by $\frac{\partial f}{\partial x_k}$, and we have $\frac{\partial f}{\partial x_k}(a) = \lim_{t \to 0} \frac{f(a_1, \ldots, a_k+t, \ldots, a_n) - f(a_1, \ldots, a_n)}{t}$.

(4.5) If function f is differentiable at $a \in \mathbb{R}^n$, then f is continuous at a.
(4.6) If function f is differentiable at $a \in \mathbb{R}^n$, then all partial derivatives exist at a and

$$\nabla f(a) = (\frac{\partial f}{\partial x_1}(a), \cdots, \frac{\partial f}{\partial x_n}(a)).$$

(4.7) Conversely, the existence of partial derivatives does not imply differentiability, not even continuity. Function g in (4.1) and (4.2) provides such an example. However, we have
(4.8) Let $f : \mathbb{R}^n \to \mathbb{R}$ be such that $\frac{\partial f}{\partial x_k}$ exist for all $k = 1, \ldots, n$ at all points and they are all continuous on $\mathbb{R}^n$. Then f is differentiable at any point and $\nabla f = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$.

Rules for differentiation:
(4.9) If f, g are differentiable at a, so are $f + g$, fg, and $\nabla(f + g) = \nabla f + \nabla g$ and $\nabla(fg) = f\nabla g + g\nabla f$. Also, f/g is differentiable at a provided $g(a) \neq 0$, and $\nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}$.

We will say that $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable at $\alpha \in \mathbb{R}$ if ϕ is differentiable in some open interval containing α and if ϕ' is continuous at α.
(4.10) (*Chain rule*) If f is differentiable at a and $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable at $f(a)$, then $\phi \circ f$ is differentiable at a and $\nabla(\phi \circ f)(a) = \phi'(f(a))\nabla f(a)$.
(4.11)(Directional derivative) Let $\nu \in \mathbb{R}^n$ be any vector. Then we define *the derivative of f in the direction ν* by $\frac{\partial f}{\partial \nu}(a) = \lim_{t \to 0} \frac{f(a+t\nu) - f(a)}{t}$, if such limit exists.
(4.12) If f is differentiable at a, then $\frac{\partial f}{\partial \nu}(a)$ exists and $\frac{\partial f}{\partial \nu}(a) = \nabla f(a) \cdot \nu$.
(4.13) (*MVT - Mean Value Theorem*) Let $x, y \in \mathbb{R}^n$ and suppose f is differentiable in some set containing the segment joining points x and y. Then there is some $\theta \in (0, 1)$ such that $f(y) - f(x) = \nabla f(x + \theta(y - x)) \cdot (y - x)$.