M3P1/M4P1 (2005) Dr M Ruzhansky
Metric and Topological Spaces

Summary of the course: definitions, examples, statements.

Chapter 1: Metric spaces and convergence.

(1.1) Recall the standard "distance function" on $\mathbb{R}$, defined by $d(x,y) = |x-y|$. This will be a model for more general "distance" or "metric" functions on other sets.

(1.2) The *Cartesian product* $X \times Y$ of two sets X and Y is defined as a set of all ordered pairs from X and Y, i.e. $X \times Y = \{(x,y) : x \in X, y \in Y\}$.

Metrics and metric spaces:
(1.3) A *metric space* (X,d) is (by definition) a set X together with a real valued function (*metric*) $d : X \times X \to \mathbb{R}$ satisfying
(M1) $d(x,y) \geq 0$ for all $x, y \in X$ (*non-negativity*); $d(x,y) = 0$ if and only if $x = y$ (*non-degeneracy*);
(M2) $d(x,y) = d(y,x)$ for all $x, y \in X$ (*symmetry*);
(M3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$ (*triangle inequality*).

Note that any set can be equipped with a so-called *discrete* metric defined by $d(x,y) = 1$ if $x \neq y$ and $d(x,y) = 0$ if $x = y$. In this way every set becomes a metric space. In general, we can put many different metrics on a set. Other examples include the following family of metrics on $\mathbb{R}^n$: $d_1(x,y) = \sum_{i=1}^n |x_i - y_i|$ (taxicab metric), $d_2(x,y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ (Euclidean metric), $d_\infty(x,y) = \max_{1\leq i\leq n} |x_i - y_i|$ (sup-metric), and $d_p(x,y) = \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}$ for all $1 \leq p < \infty$.

(1.4) (*Spaces of bounded and continuous functions*) Let $a < b$. Then we define

$$B([a,b]) = \{f : [a,b] \to \mathbb{R} : f \text{ is bounded}\},\ C([a,b]) = \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}.$$

They become metric spaces when equipped with "sup-norm" (or "*sup-metric*") $d_\infty(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$. Note that on $C([a,b])$ we may also define another metric by $d_1(f,g) = \int_a^b |f(x) - g(x)| dx$. Check that this is no longer a metric on $B([a,b])$ since the non-degeneracy property (M1) in (1.3) fails. There are, of course, other possible metrics on these spaces. For example, for every $p \in [1,\infty)$, $d_p(f,g) = \left(\int_a^b |f(x) - g(x)|^p dx\right)^{1/p}$ is a metric on $C([a,b])$.

(1.5) (*Induced metrics*) If (X,d) is a metric space and $A \subset X$ is a subset of X, then the restriction d_A of d to $A \times A$ is a metric "*induced*" on A from X. Thus, (A, d_A) is a metric space and A is called a *metric subspace* of X. When talking about a subset of a metric space we always equip it with the induced metric (unless we state otherwise).

(1.6) (*Balls*) Let (X,d) be a metric space, $a \in X$ and $r > 0$. The *"open" ball* $B_r(a)$ centred at a with radius r is defined by $B_r(a) = \{x \in X : d(x,a) < r\}$.

Bounded sets:
(1.7) A subset S of a metric space (X,d) is called *bounded* if there exists $r > 0$ and $a \in X$ such that $S \subset B_r(a)$.
(1.8) Suppose that S is a bounded subset of (X,d) and $c \in X$. Then $S \subset B_R(c)$ for some $R > 0$. This property shows that we can change the centre of the ball in definition (1.7).
(1.9) If $S_1, \ldots, S_n$ are bounded subsets of (X,d) and $X \neq \emptyset$, then the union $S_1 \cup \cdots \cup S_n$ is a bounded subset of (X,d).
Note that infinite unions of bounded sets do not have to be bounded. Also note, that all sets

are bounded in a space with the discrete metric since balls with radii strictly larger than one are equal to the whole space. Of course, this situation is quite exceptional.

Convergence in metric spaces:
(1.10) We say that a sequence x_n of points in a metric space (X, d) *converges* to $x \in X$ if $d(x_n, x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$. We have the following simple characterization of convergence:
(1.11) $x_n \to x \iff \forall \epsilon > 0\ \exists N\ \forall n > N\ d(x_n, x) < \epsilon \iff \forall \epsilon > 0\ \exists N\ \forall n > N\ x_n \in B_\epsilon(x)$.
All these are also equivalent to saying that any ball centred at x contains all but finitely many of x_n's.
(1.12) (*Uniqueness of limits*) Let (X, d) be a metric space. Suppose that $x_n \to x$ and $x_n \to y$. Then $x = y$.

Chapter 2: Open sets and topological spaces.

Let (X, d) be a metric space. Then we have the following definitions and properties:

(2.1) A set $U \subset X$ is called *open* if for every $x \in U$ there is an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. An open set containing x is called *an (open) neighbourhood of x*. We have the following simple properties:
(2.2) Every "open" ball $B_r(a)$ is open in (X, d). This justifies the terminology in (1.6).
(2.3) $x_n \to x$ if and only if every open neighborhood of x contains all but finitely many of x_n's.

Suppose I is any set. Assume that for each $i \in I$ we are given a set A_i. Then I is called an *index set* for the collection of sets A_i.
(2.4) We have the following properties of open sets:
(T1) $\emptyset$ and X are open;
(T2) The union of any collection of open subsets of X is open;
(T3) The intersection of any finite collection of open subsets of X is open.

Continuous mappings:
Let (X_1, d_1), (X_2, d_2) be metric spaces and let $f : X_1 \to X_2$.
(2.5) Let $a \in X_1$. Then f is said to be *continuous at a* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $d_1(x, a) < \delta$ implies $d_2(f(x), f(a)) < \epsilon$. If f is continuous at all points of X_1, it is said to be *continuous* (*on* X_1).
Let $S \subset X_2$. The *preimage* of S under f is defined by $f^{-1}(S) = \{x \in X_1 : f(x) \in S\}$.
(2.6) The following statements are equivalent:
(i) f is continuous on X_1;
(ii) $\forall a \in X_1\ \forall$ ball $B_\epsilon(f(a))\ \exists$ ball $B_\delta(a)$ such that $B_\delta(a) \subset f^{-1}(B_\epsilon(f(a)))$;
(iii) for every open $U \subset X_2$ the preimage $f^{-1}(U)$ is open in X_1.

Topological spaces:
(2.7) Let X be a set. A *topology* on X is (by definition) any collection T of subsets of X satisfying
(T1) $\emptyset, X \in T$;
(T2) Any union of sets in T is in T;
(T3) Any finite intersection of sets in T is in T.
The sets in T are called *open sets* of the topology. A *topological space* is (by definition) a set together with a topology on it.

If (X, d) is a metric space, the collection of all sets open according to definition (2.1) is called the *metric topology* (for the metric d), and will be denoted by $T(d)$. Thus, (2.8) Every metric space is a topological space. By default, it will be always equipped with the metric topology $T(d)$.

Let X be a set. Other examples of topologies are the *discrete topology* consisting of all subsets of X and the *indiscrete* (or co-finite) topology consisting of the empty set and all $U \subset X$ such that $X \backslash U$ is finite. Note that the discrete topology is the metric topology for the *discrete* metric d defined by $d(x, y) = 1$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$. Also note that the indiscrete topology is never a metric topology if X is infinite (since any sequence of different points converges to any point of X, if we use (2.9). This contradicts the uniqueness of limits property (1.12) of metric spaces).

(2.9) (*Convergence in topological spaces*) We say that $x_n \to x$ if every open set containing x contains all but finitely many of x_n's.

By characterization (2.3) this notion of convergence coincides with the one for metric spaces, defined in (1.10).

Note that in general, in topological spaces limits are not unique. For example, if $X = \{1, 2\}$ and $T = \{\emptyset, \{1\}, \{1, 2\}\}$, then the sequence $x_n \equiv 1$ satisfies $x_n \to 1$ and $x_n \to 2$. This is not possible in metric spaces (and in more general Hausdorff spaces from Chapter 3) because of the uniqueness of limits property (1.12).

Equivalent metrics:

(2.10) It may happen that two metrics d_1 and d_2 on a set X define the same topology, i.e. $T(d_1) = T(d_2)$. Then these metrics are called *equivalent*. We have the following criteria:

(2.11) Let d_1 and d_2 be two metrics on X such that for some $K > 0$ we have $d_2(x, y) \leq K d_1(x, y)$ for all $x, y \in X$. Then every d_2-open set is d_1-open, or $T(d_2) \subset T(d_1)$.

(2.12) Let T_1, T_2 be two topologies on a set X. Topology T_1 is said to be *stronger* than T_2 if $T_2 \subset T_1$ (so T_1 contains more sets than T_2). In this case T_2 is said to be *weaker* than T_1. In (2.11) we have $T(d_2) \subset T(d_1)$, so the topology defined by d_2 is weaker than the topology defined by d_1.

Note that on a set X, the discrete topology is the strongest topology and $\{\emptyset, X\}$ is the weakest one.

As a consequence of (2.11), we get the following criterion of the equivalence of two metrics:

(2.13) Let d_1 and d_2 be two metrics on X such that for some $K, H > 0$ we have $d_2(x, y) \leq K d_1(x, y)$ and $d_1(x, y) \leq K d_2(x, y)$, for all $x, y \in X$. Then metrics d_1 and d_2 are equivalent, i.e. $T(d_1) = T(d_2)$.

As an example we can conclude that all metrics d_p, $1 \leq p \leq \infty$, on $\mathbb{R}^n$, are equivalent. Indeed, it is easy to check that $d_\infty(x, y) \leq d_p(x, y) \leq n^{1/p} d_\infty(x, y)$ for all $x, y \in \mathbb{R}^n$ and all $1 \leq p < \infty$. Hence $T(d_p) = T(d_\infty)$ for $1 \leq p < \infty$ and so all these metrics are equivalent and lead to the same (Euclidean) topology. On $C([a, b])$, we have $d_1(f, g) \leq (b - a) d_\infty(f, g)$, so $T(d_1)$ is weaker than $T(d_\infty)$.

Continuous mappings in topological spaces:

(2.14) A mapping $f : X_1 \to X_2$ between two topological spaces is called *continuous* if for every U open in X_2 its preimage $f^{-1}(U)$ is open in X_1.

By (2.6) this notion is consistent with the notion of continuity in metric spaces, which was defined in (2.5). Note that we define continuity on the whole space only, not at particular points, as in metric spaces in (2.5).

(2.15) Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous mappings. Then the composition $g \circ f : X_1 \to X_3$ is continuous. Here we use the usual definition $(g \circ f)(x) = g(f(x))$.

(2.16) Let X and Y be two topological spaces. A mapping $f : X \to Y$ is called a *homeomorphism* if

(i) $f : X \to Y$ is bijective (i.e. injective and surjective, or one-to-one and onto);
(ii) $f : X \to Y$ is continuous;
(iii) the inverse mapping $f^{-1} : Y \to X$ is continuous.
Topological spaces X and Y are said to be *homeomorphic* if there exists a homeomorphism $f : X \to Y$. Homeomorphic spaces are sometimes called *topologically equivalent*. For example, intervals $(0, 1)$ and (a, b) are topologically equivalent, for any $a < b$. Indeed, function $f : (0, 1) \to (a, b)$ defined by $f(x) = a + (b - a)x$ is a homeomorphism between them.
(2.17) A property which holds in all topologically equivalent spaces is called a *topological property*.

Such properties will be of high importance because they are invariants of topological spaces. Using these properties we can decide when two spaces are not topologically equivalent. Examples of topological properties are "X has n elements", "infinite X is countable", "all subsets of X are open", etc. This shows, for example, that $\mathbb{Q}$ and $\mathbb{R}$ are not topologically equivalent whatever topologies we put on them; or that $(\mathbb{R}, \text{discrete})$ and $(\mathbb{R}, \text{Euclidean})$ are not topologically equivalent.
Note that property (2.15) and definitions (2.16) and (2.17) imply that the topological equivalence is an equivalence relation on the set of all topological spaces.
The following is another characterisation of equivalent metrics in terms of topologies:
(2.18) Let d_1 and d_2 be two metrics on X. Then d_1 and d_2 are equivalent if and only if the identity map from (X, d_1) to (X, d_2) is a homeomorphism.

Closed sets:
(2.19) A subset A of a topological space X is called *closed* if its complement $X \backslash A$ is open.
(2.20) Let X be a topological space. Then
(i) $\emptyset$ and X are closed;
(ii) any intersection of closed sets is closed;
(iii) any <u>finite</u> union of closed sets is closed.
(2.21) The *closure* in X of a subset $A \subset X$ is defined as the intersection of all closed sets in X containing A. The closure of A is denoted by $\bar{A}$. The *interior* $\text{int}A$ of A is defined as the union of all open sets contained in A. The *boundary* ∂A of A is defined by $\bar{A} \backslash \text{int}A$. It follows from (2.20), part (ii), and property (T2) that the closure $\bar{A}$ is closed and the interior $\text{int}A$ is open.
(2.22) Let X be a topological space and let $S \subset X$. Then the boundary ∂S of S is always closed. Further, let open U and closed F be such that $U \subset S \subset F$. Then $U \subset \text{int}S$ and $\bar{S} \subset F$.
(2.23) Let X be a topological space and let $S \subset X$. Then S is open if and only if for every $x \in S$ there is an open set U_x such that $x \in U_x$ and $U_x \subset S$.
Compare (2.23) with definition (2.1) of open sets in metric spaces.
(2.24) Let (X, T) and (Y, S) be topological spaces and let $f : X \to Y$ be continuous. If $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.
Recall that this was a characterisation of continuity of functions $f : \mathbb{R} \to \mathbb{R}$ at the very beginning of M2P1.

Chapter 3: Hausdorff spaces and subspaces.

Topological subspaces:
(3.1) Let (X, T) be a topological space and let $Y \subset X$. We define the *induced topology* on Y by $T_Y = \{U \cap Y : U \in T\}$. Then we have:
(3.2) (Y, T_Y) as above is a topological space.
Unless stated otherwise, we will always equip subspaces of a topological space with the induced topology and call them *topological subspaces*.

The following statement says that in metric spaces, the topology defined by the induced metric is the same as the induced metric topology:
(3.3) Let (X, d) be a metric space, let $Y \subset X$, and let $d_Y = d|_{Y \times Y}$ be the induced metric. Then $T(d)_Y = T(d_Y)$.
(3.4) Let (X, T) be a topological space and let $Y \subset X$. Let $i_Y : Y \to X$ be the injection mapping: $i_Y(y) = y$ for all $y \in Y$. Then i_Y is continuous from (Y, T_Y) to (X, T).

Product spaces:
(3.5) Let (X_1, T_1) and (X_2, T_2) be topological spaces. A subset of $X_1 \times X_2$ is said to be *open in the product topology* if it is a union of sets of the form $U_1 \times U_2$, where U_i is an open subset of X_i, $i = 1, 2$.
(3.6) The above definition makes $X_1 \times X_2$ a topological space. This topology is called the *product topology.*
(3.7) The product topology on $\mathbb{R} \times \mathbb{R}$ is the usual (metric) topology of $\mathbb{R}^2$. By induction this can be easily extended to $\mathbb{R}^n$. Note that in the proof of (3.7) we establish an interesting fact that open sets in the Euclidean $\mathbb{R}$ are unions of open intervals.

Hausdorff spaces:
Consider an example of $X = \{1, 2, 3\}$ with topology $T = \{\emptyset, \{1\}, \{1, 2\}, X\}$. Let $x_n = 1$ for all n. Then $x_n \to 1$, $x_n \to 2$ and $x_n \to 3$. Also, any sequence of different element of an infinite set X converges to any element of X when X is equipped with the indiscrete topology. To eliminate these phenomena, one defines the notion of a Hausdorff space.
(3.8) A topological space (X, T) is called *Hausdorff* if for every $a, b \in X$ with $a \neq b$, there exist open sets U, V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.
(3.9) Every metric space is Hausdorff.
(3.10) In a Hausdorff space, every convergent sequence has a unique limit.
(3.11) All one-element sets in a Hausdorff spaces are closed. Therefore, all finite sets are also closed. In particular, if (X, T) is a finite Hausdorff space, then the topology T is discrete.
(3.12) Every topological subspace of a Hausdorff space is Hausdorff.
(3.13) Let (X_1, T_1) and (X_2, T_2) be topological spaces and let $f : X_1 \to X_2$ be injective (i.e. one-to-one) and continuous. If X_2 is Hausdorff then X_1 is Hausdorff. Thus,
(3.14) "Hausdorff" is a topological property.
(3.15) If X and Y are Hausdorff then $X \times Y$ is also Hausdorff (in the product topology).

Chapter 4: Compact spaces.

Let (X, T) be a topological space.
(4.1) A collection $\{U_\alpha\}$, $\alpha \in I$, of subsets of X is called *an open covering* of X if each U_α is open and $\cup_{\alpha \in I} U_\alpha = X$.
(4.2) Topological space X is called *compact* if every open covering of X has a finite subcovering.
(4.3) (*Heine-Borel theorem*) Closed intervals $[a, b] \subset \mathbb{R}$ are compact (in the standard Euclidean metric topology).
(4.4) Let X be compact and let $f : X \to \mathbb{R}$ be continuous. Then f is bounded (i.e. $f(X)$ is a bounded set in Euclidean $\mathbb{R}$).
Note that combined, (4.3) and (4.4), recover the corresponding statement from M2P1.

Compact subspaces:
(4.5) Let (X, T) be a topological space and let $A \subset X$. Then the topological space (A, T_A) is compact if and only if for every collection $C = \{U_\alpha\}$ of open sets $U_\alpha \subset X$ in (X, T) for which $A \subset \bigcup_\alpha U_\alpha$, there exists a finite subcollection of C which still covers A.
(4.6) A compact subspace of a Hausdorff space is closed.

(4.7) A closed subspace of a compact topological space is compact.
(4.8) Thus, if X is a compact Hausdorff space and $A \subset X$, then A is compact if and only if A is closed.
(4.9) Let X be compact and $f : X \to Y$ be continuous. Then the image set $f(X)$ is compact (in the induced topology of Y). Thus,
(4.10) "Compactness" is a topological property.
(4.11) A compact subspace of a metric space is bounded.
(4.12) Thus, a compact subset of any metric space is closed and bounded.
(4.13) If X and Y are compact topological spaces, then $X \times Y$ is compact (in the product topology).
(4.14) A subset of $\mathbb{R}^n$ is compact if and only if it closed and bounded.
In particular, this implies the Heine-Borel theorem (4.3). However, do not forget that in lectures we actually used (4.3) and properties of compact sets in the product topology to obtain (4.14).

Sequences and compactness:
Recall the *Bolzano-Weierstrass* theorem from M1P1: any bounded sequence of real numbers has a convergent subsequence. A sequence of points x_n is bounded means that $x_n \in [a, b]$ for some $[a, b]$. So this theorem can be reformulated in the following way:
(4.15) Let $X = [a, b] \subset \mathbb{R}$. Then any sequence of points of X has a convergent subsequence.
This suggests the following definition:
(4.16) A topological space X is called *sequentially compact* if every sequence of points of X has a convergent subsequence.
The first main theorem is:
(4.17) If X is a compact <u>metric</u> space, it is sequentially compact.
(4.18) (FIP: finite intersection property) X is compact if and only if the closed sets in X have the finite intersection property, i.e. if $\{C_\alpha\}$ is a collection of closed sets in X with $\bigcap_\alpha C_\alpha = \emptyset$, then there is some finite subcollection $C_1, C_2, \ldots, C_n$ of $\{C_\alpha\}$ such that $\bigcap_{i=1}^n C_i = \emptyset$.
(4.19) Let $A \subset X$. Then $x \in \bar{A}$ if and only if every open set containing x contains a point of A.
We also have the converse of (4.17):
(4.20) If X is a sequentially compact <u>metric</u> space, then X is compact.
(4.21) Thus, if (X, T) is a topological space with T being the metric topology for some metric, then X is compact if and only if it is sequentially compact.
In general (non-metrisable) topological spaces compactness and sequential compactness are not equivalent.
The following properties of sequentially compact metric spaces are useful.
(4.22) (Lebesgue's covering lemma) Let X be sequentially compact and let C be an open covering of X. Then there is an $\epsilon > 0$ such that every ball of radius ϵ is contained in some set $U \in C$. Such ϵ is called a Lebesgue number of the covering C.
(4.23) If X is sequentially compact, it is *totally bounded*, i.e. for every $\epsilon > 0$ there is a finite collection of balls $B_\epsilon(x_i)$ with radius ϵ that covers X.

Chapter 5: Complete metric spaces.

A *Cauchy sequence* in a metric space (X, d) is (by definition) a sequence $x_n \in X$ such that for every $\epsilon > 0$ there is N such that for all $n, m > N$ holds $d(x_n, x_m) < \epsilon$. Here are some basic properties of Cauchy sequences:
(5.1) Any Cauchy sequence is bounded.
(5.2) If a Cauchy sequence x_n has a convergent subsequence, say $x_{n_i} \to x \in X$, then $x_n \to x$.
(5.3) Any convergent sequence is a Cauchy sequence.

Complete metric spaces:
(5.4) Metric space X is called *complete* if every Cauchy sequence of points of X converges to a point of X.
We have the following properties:
(5.5) Every compact metric space is complete.
(5.6) The real line $\mathbb{R}$ with the Euclidean metric is complete.
(5.7) $\mathbb{R}^n$ is complete with respect to any of the equivalent metrics d_p, $1 \leq p \leq \infty$.
(5.8) Every closed subset of a complete metric space is complete.

Space $C([a, b])$ of continuous functions:
(5.9) Let $f_n : [a, b] \to \mathbb{R}$ be a family of functions. We say that a sequence f_n converges to $f : [a, b] \to \mathbb{R}$ *pointwise* if $f_n(x) \to f(x)$ for every $x \in [a, b]$. In other words, it means that for every $x \in [a, b]$ and every $\epsilon > 0$ there exists $N = N(\epsilon, x)$ such that for all $n > N$ we have $|f_n(x) - f(x)| < \epsilon$.
(5.10) Let $f, f_n : [a, b] \to \mathbb{R}$. We say that f_n converges to f *uniformly* if for every $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for all $n > N$ and all $x \in [a, b]$ we have $|f_n(x) - f(x)| < \epsilon$.

The difference between these two definitions is that in (5.9) the number N depends on ϵ and x while in (5.10) it is independent of x (think about this important difference!) Now, we have the following properties of continuous functions:
(5.11) Let $f_n : [a, b] \to \mathbb{R}$ be a family of continuous functions. If f_n converges to $f : [a, b] \to \mathbb{R}$ uniformly on $[a, b]$ then f is continuous.
(5.12) (recall definition (1.4)) Let $C([a, b])$ be the space of all continuous $f : [a, b] \to \mathbb{R}$. We equip $C([a, b])$ with the sup-metric d_∞, defined by (1.4). Then we have:
(5.13) $f_n \to f$ in the metric space $(C([a, b]), d_\infty)$ if and only if f_n converges to f uniformly on $[a, b]$.
(5.14) Space $C([a, b])$ with the sup-metric is a complete metric space.

Chapter 6: Fixed point theorem.

(6.1) Let (X, d) be a metric space. A mapping $f : X \to X$ is called a *contraction* if there is k, $0 \leq k < 1$, such that for every $x, y \in X$ we have $d(f(x), f(y)) \leq kd(x, y)$.
(6.2) If (X, d) is a non-empty complete metric space, then every contraction mapping f on X has a unique fixed point, i.e. there is a unique point $x \in X$ such that $f(x) = x$.
(6.3) (Application: Fredholm integral equations) Let $p : [0, 1] \to \mathbb{R}$ be continuous, $p \geq 0$, and $\int_0^1 p(t)dt < 1$. Let $F \in C([0, 1])$. Then there exists a unique function $f \in C([0, 1])$ such that $f(x) = F(x) - \int_0^x f(t)p(t)dt$.
With this fact we can argue the existence of a solution f to the differential equation $f' + pf = F$.

Chapter 7: Connected topological spaces.

Connectedness:
(7.1) Let (X, T) be a topological space. Then the following three statements are equivalent:
(i) There exist non-empty open subsets U and V of X such that $U \cap V = \emptyset$ and $U \cup V = X$.
(ii) There exists a subset U of X such that U is open and closed, and $U \neq \emptyset$ and $U \neq X$.
(iii) There is a continuous function from X <u>onto</u> $\{0, 1\}$, where $\{0, 1\}$ is equipped with the discrete topology.

This property is taken as the definition of a disconnected topological space:
(7.2) A topological space (X, T) is called *disconnected* if there exist non-empty open sets U and V such that $U \cap V = \emptyset$ and $U \cup V = X$. Otherwise X is called *connected*.

Note as an example, that the only connected sets in X with discrete topology are the empty set and one-point sets. Of course, this situation is quite special.
(7.3) Let (X_1, T_1), (X_2, T_2) be topological spaces and let $f : X_1 \to X_2$ be continuous. If X_1 is connected, then $f(X_1)$ is also connected (as a topological subspace of (X_2, T_2)). Thus,
(7.4) "Connectedness" is a topological property.
(7.5) Let (X, T) be a topological space and let $S \subset X$. If S is connected, then its closure $\bar{S}$ is also connected.

Path-connectedness:
(7.6) Suppose that a and b are two points in a topological space (X, T). A *path* from a to b is (by definition) a continuous mapping $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. Here $[0, 1]$ is equipped with the standard Euclidean topology.
(7.7) A topological space (X, T) is called *path-connected* if for any two elements $a, b \in X$ there exists a path from a to b. Then we have the following theorem:
(7.8) Every path-connected topological space is connected.

(7.9) (*Glueing of paths*) Let $f, g : [0, 1] \to X$ be two continuous mappings such that $f(0) = a, f(1) = b, g(0) = b, g(1) = c$. Define $h : [0, 1] \to X$ by $h(t) = f(2t)$ for $0 \leq t \leq 1/2$ and by $h(t) = g(2t - 1)$ for $1/2 \leq t \leq 1$. Then $h(0) = a$, $h(1) = c$, and h is continuous, thus defining a path from a to c, which is called the *glueing* of paths f and g. Using this notion, we can prove the following theorem:
(7.10) Every open connected subset S of $\mathbb{R}^n$ with the standard topology is path-connected.
In general topological spaces, the converse to (7.8) is not true: connectedness does not imply path-connectedness.

Special case: intervals in Euclidean $\mathbb{R}$:
(7.11) An *interval* is (by definition) a subset of $\mathbb{R}$ of one of the following forms $(a, b), (a, b], [a, b)$, $[a, b]$, $(-\infty, b)$, $(-\infty, b], (a, \infty), [a, \infty), (-\infty, \infty)$. Then we have:
(7.12) Every interval in $\mathbb{R}$ with the Euclidean topology is connected.
(7.13) A subset S of $\mathbb{R}$ is an interval if and only if for any $a, c \in S$ and any $b \in \mathbb{R}$ such that $a < b < c$ it follows that $b \in S$.
(7.14) Let S be a subset of $\mathbb{R}$. If S is not an interval, then it is disconnected.
(7.15) The connected subsets of $\mathbb{R}$ with the Euclidean topology are the intervals.

Chapter 8: Completions (of metric spaces).

This chapter is a continuation of Chapter 5.
(8.1) Let (X, d) be a metric space. A complete metric space X^* is called a *completion* of X if X is a topological subspace of X^* and $\bar{X} = X^*$.
For example, $\mathbb{R}$ is a completion of the set $\mathbb{Q}$ of rational numbers. The main theorem here is
(8.2) Every metric space (X, d) has a completion. This completion is unique up to an isometry leaving X fixed.