Analytic Methods in Partial Differential Equations

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Some exercises to the course

1. Work out the details of the proof of the Riemann–Lebesgue theorem. In particular, investigate the case when some of ξ_j's are zero.

2. Prove that $f \in \mathcal{S}(\mathbb{R}^n)$ if and only if for all $\alpha \geq 0$ and $N \geq 0$ there is a constant $C_{\alpha,N}$ such that $|\partial^\alpha \varphi(x)| \leq C_{\alpha,N}(1+|x|)^{-N}$ for all $x \in \mathbb{R}^n$.

3. Prove that $\int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^\rho} < \infty$ if and only if $\rho > n$. Also prove that $\int_{|x|\leq 1} \frac{dx}{|x|^\rho} < \infty$ if and only if $\rho < n$.

4. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$. Prove that $\widehat{\varphi\psi}(\xi) = (\widehat{\varphi} * \widehat{\psi})(\xi)$.

5. Prove the following generalisation of Hölder's inequality. Let $1 \leq p, q, s \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{s}$. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$. Prove that $fg \in L^s(\mathbb{R}^n)$ and that $||fg||_{L^s} \leq ||f||_{L^p}||g||_{L^q}$.

6. Let f be a smooth function such that f and all of its derivatives are bounded by some polynomials. Prove that the mapping $u \mapsto fu$ is well-defined and continuous from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$.

7. Prove that $\widehat{1} = \delta$.

8. Work out the details of all the statements from 1.3.9 about distributions.

9. Let Ω be an open subset of $\mathbb{R}^n$. Show that our canonical identification of functions with distributions yields the inclusions $L^p_{loc}(\Omega) \subset \mathcal{D}'(\Omega)$ for all $1 \leq p \leq \infty$. Prove that these mappings $f \mapsto u_f$ are continuous from $L^p_{loc}(\Omega)$ to $\mathcal{D}'(\Omega)$.

10. Define $u : \mathbb{R} \to \mathbb{R}$ by
$$u(x) = \begin{cases} x, & \text{if} \quad x \leq 1, \\ 2, & \text{if} \quad x > 1. \end{cases}$$
Calculate its distributional derivative.

11. Prove that the δ-distribution is not an element of $L^1_{loc}(\mathbb{R}^n)$.

12. Define $u(x) = |x|^{-a}$ for $x \in B(0,1) \subset \mathbb{R}^n$, $x \neq 0$. Also set $u(0) = 0$. Find conditions on a, n, p, k for which $u \in L^p_k(B(0,1))$.

13. Work out the details of 1.3.15 about mollifiers.

14. Let T_a be a pseudo-differential operator with symbol $a \in S^m$. Let $f \in \mathcal{S}(\mathbb{R}^n)$. Show that all the derivatives of $T_a f$ are rapidly decreasing, thus completing the proof of 2.1.3 that $T_a f \in \mathcal{S}(\mathbb{R}^n)$.

15. Work out all the details of the convergence criterion in 2.1.3.

16. Let $a \in S^m$ and let $\gamma \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\gamma = 1$ near the origin. For $\epsilon > 0$ define $a_\epsilon(x,\xi) = a(x,\xi)\gamma(\epsilon x, \epsilon\xi)$. Prove that $a_\epsilon \in S^m$ uniformly in $0 < \epsilon \leq 1$ (i.e. show that the constants in symbolic inequalities may be chosen independent of $0 < \epsilon \leq 1$); Prove that $\partial_x^\alpha \partial_\xi^\beta a_\epsilon(x,\xi) \to \partial_x^\alpha \partial_\xi^\beta a(x,\xi)$ as $\epsilon \to 0$, uniformly in $0 < \epsilon \leq 1$, for all $x, \xi \in \mathbb{R}^n$.

17. Let $a \in S^m$. Prove that the adjoint operator $T_a^* : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is well-defined and continuous.

18. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Prove that all the derivatives of $(u * \varphi)(x) = u(\tau_x R\varphi)$ with respect to x are continuous.

19. In the proof of the composition formula in the first case, show that the derivatives of the error have the symbolic behaviour. Namely, let $a \in S^{m_1}$, $b \in S^{m_2}$, and assume that $b(y,\xi)$ is compactly supported with respect to y. Let $R_N(x,y,\xi) = a(x,\xi+\eta) - \sum_{|\alpha|<N} \frac{(2\pi i)^{-|\alpha|}}{\alpha!} \partial_\xi^\alpha a(x,\xi) \partial_x^\alpha b(x,\xi)$ be the remainder in the Taylor's formula. Prove that
$$\left| \partial_x^\beta \partial_\xi^\gamma \left(\int_{\mathbb{R}^n} e^{2\pi i x\cdot\eta} R_n(x,\xi,\eta) \widehat{b}(\eta,\xi) d\eta \right) \right| \leq C_{\beta,\gamma,N} (1+|\xi|)^{m_1+m_2-N-|\gamma|},$$
for all $x, \xi \in \mathbb{R}^n$ and all multi-indices β, γ.

20. Let $x_0 \in \mathbb{R}^n$, $a \in S^{m_1}$, $b_2 \in S^{m_2}$, and assume that $b_2(y,\xi) = 0$ for $|y - x_0| < 1$. Let
$$c(x,\xi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{2\pi i (x-y)\cdot(\eta-\xi)} a(x,\eta) b_2(y,\xi) dy d\eta.$$
Prove that
$$\left| \partial_x^\beta \partial_\xi^\gamma c(x,\xi) \right| \leq C_{\beta,\gamma,N} (1+|\xi|)^{m_1+m_2-N},$$
for all $|x - x_0| \leq 1/2$, all $\xi \in \mathbb{R}^n$, all multi-indices β, γ, and all $N \geq 0$.

21. Work out the details of the proof of the theorem that says that an operator with a compound symbols is a pseudo-differential operator. In particular, work out the part with the estimation of the remainder, and the part when the compound symbols in not compactly supported.

22. Prove that if a_k is a smooth function such that $a_k(x,\lambda\xi) = \lambda^k a_k(x,\xi)$ for $\lambda > 1$ and $|\xi| \geq 1$, then $a_k \in S^k$.

23. Prove that the composition of two pseudo-differential operators with classical symbols is again a pseudo-differential operator with a classical symbol.