
Exercise 1.1. Consider the meromorphic function

$$F(z) = \frac{1}{1+z^2} \frac{\cos\left[(\pi-\theta)z\right]}{2\sin \pi z}$$

a) Show that the poles of $F(z)$ occur at $z = \pm i$ and $z = n$, for $n \in \mathbb{Z}$, with:

$$\mathrm{Res}[F, \pm i] = -\frac{\cosh(\pi-\theta)}{4\sinh \pi}, \qquad \mathrm{Res}[F, n] = \frac{\cos(n\theta)}{2\pi(1+n^2)}.$$

b) For $N \in \mathbb{N}$, let $\Gamma_i^{(N)}$ be the curves

$$\Gamma_1^{(N)}(t) = \left(N + \frac{1}{2}\right)(1+it), \qquad -1 \le t \le 1$$

$$\Gamma_2^{(N)}(t) = \left(N + \frac{1}{2}\right)(-t+i), \qquad -1 \le t \le 1$$

$$\Gamma_3^{(N)}(t) = \left(N + \frac{1}{2}\right)(-1-it), \qquad -1 \le t \le 1$$

$$\Gamma_4^{(N)}(t) = \left(N + \frac{1}{2}\right)(t-i), \qquad -1 \le t \le 1$$

and let $\Gamma^{(N)}$ be the closed contour which results from following Γ_1, Γ_2, Γ_3 and Γ_4 in turn. Sketch $\Gamma^{(N)}$, together with the locations of the poles of F in the complex plane.

c) Show that if $0 \le \theta \le 2\pi$, then:

$$\left| \int_{\Gamma^{(N)}} F(z)dz \right| \le \frac{C}{N},$$

where C is a fixed constant, independent of N.

d) By applying Cauchy's residue theorem, show that for $0 \le \theta \le 2\pi$:

$$\left| \frac{\cosh(\pi-\theta)}{2\sinh \pi} - \sum_{n=-N}^{N} \frac{e^{in\theta}}{1+n^2} \right| \le \frac{C}{N},$$

and conclude that

$$\frac{\cosh(\pi-\theta)}{2\sinh \pi} = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1+n^2},$$

with the sum converging uniformly in θ.

Exercise 1.2. Suppose that $f \in C^0(\mathbb{R})$ is a continuous function with period 2π, i.e. $f(\theta) = f(\theta + 2\pi)$. For $\theta \in [0, 2\pi)$, define:

$$\psi(\theta) := \int_0^\theta f(\alpha) \frac{\cosh(\pi - \theta + \alpha)}{2 \sinh \pi} d\alpha + \int_\theta^{2\pi} f(\alpha) \frac{\cosh(-\pi - \theta + \alpha)}{2 \sinh \pi} d\alpha$$

and extend ψ to a function on $\mathbb{R}$ by periodicity: $\psi(\theta) = \psi(\theta + 2\pi)$.

a) Show that $\psi \in C^0(\mathbb{R})$.

b) By directly differentiating the formula, show that:

$$\psi'(\theta) = -\int_0^\theta f(\alpha) \frac{\sinh(\pi - \theta + \alpha)}{2 \sinh \pi} d\alpha - \int_\theta^{2\pi} f(\alpha) \frac{\sinh(-\pi - \theta + \alpha)}{2 \sinh \pi} d\alpha$$

and show that $\psi \in C^1(\mathbb{R})$.

c) Differentiating again, show that

$$\psi''(\theta) = -f(\theta) + \psi(\theta)$$

Conclude that $\psi \in C^2(\mathbb{R})$ is a solution to

$$-\psi''(\theta) + \psi(\theta) = f(\theta), \qquad \theta \in \mathbb{R}.$$

Exercise 1.3. Let $\phi \in C^2(\mathbb{R})$ be 2π-periodic, i.e. $\phi(\theta) = \phi(\theta + 2\pi)$, and suppose that ϕ satisfies:

$$-\phi''(\theta) + \phi(\theta) = 0, \qquad \theta \in \mathbb{R}.$$

a) Show that if ϕ attains a maximum at $\theta_0 \in \mathbb{R}$, then $\phi(\theta_0) \leq 0$.

b) Show that if ϕ attains a minimum at $\theta_0 \in \mathbb{R}$, then $\phi(\theta_0) \geq 0$.

c) Show that $\phi \equiv 0$.

d) Conclude that there is at most one $\psi \in C^2(\mathbb{R})$ satisfying $\psi(\theta) = \psi(\theta + 2\pi)$ and solving:

$$-\psi''(\theta) + \psi(\theta) = f(\theta), \qquad \theta \in \mathbb{R},$$

where $f \in C^0(\mathbb{R})$ is a given 2π-periodic function.

Exercise 1.4. For $t \in \mathbb{R}$ let:

$$\chi(t) = \begin{cases} 0 & t \leq 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

a) Show that $\chi \in C^\infty(\mathbb{R})$.

b) Show that there exists a function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

i) $0 \leq \psi \leq 1$

ii) $\operatorname{supp}\psi \subset B_2(0)$

iii) $\psi(x) = 1$ for $|x| \leq 1$.

Hint: First construct a positive smooth function $\tilde{\chi} : \mathbb{R} \to [0,1]$ such that

$$\tilde{\chi}(t) = \begin{cases} 0 & t < -1 \\ 1 & t > 1 \end{cases}$$

Exercise 1.5. a) Suppose $\phi \in \mathscr{E}(\mathbb{R}^n)$. Let $\{x_l\}_{l=1}^{\infty} \subset \mathbb{R}^n$ be a sequence with $x_l \to 0$. Show that

$$\tau_{x_l}\phi \to \phi, \qquad \text{as } l \to \infty.$$

in $\mathscr{E}(\mathbb{R}^n)$, where τ_x is the translation operator defined in equation (1.2).

b) Suppose $\phi \in \mathscr{E}(\mathbb{R}^n)$, show that

$$\Delta_i^h \phi \to D_i\phi, \qquad \text{as } h \to 0,$$

in $\mathscr{E}(\mathbb{R}^n)$, where Δ_i^h is the difference quotient defined in Example 2.

Exercise 1.6. a) Show that $\mathscr{S}$ is a vector subspace of $\mathscr{E}(\mathbb{R}^n)$. Show that if $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of rapidly decreasing functions which tends to zero in $\mathscr{S}$, then $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$.

b) Show that $\mathscr{D}(\mathbb{R}^n)$ is a vector subspace of $\mathscr{S}$. Show that if $\{\phi_j\}_{j=1}^{\infty}$ is a sequence of compactly supported functions which tends to zero in $\mathscr{D}(\mathbb{R}^n)$ then $\phi_j \to 0$ in $\mathscr{S}$.

c) Give an example of a sequence $\{\phi_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n)$ such that

i) $\phi_j \to 0$ in $\mathscr{S}$, but ϕ_j has no limit in $\mathscr{D}(\mathbb{R}^n)$.

ii) $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$, but ϕ_j has no limit in $\mathscr{S}$.

Exercise 1.7. a) Suppose $\phi \in \mathscr{S}$. Let $\{x_l\}_{l=1}^{\infty} \subset \mathbb{R}^n$ be a sequence with $x_l \to 0$. Show that

$$\tau_{x_l}\phi \to \phi, \qquad \text{as } l \to \infty.$$

in $\mathscr{S}$, where τ_x is the translation operator defined in equation (1.2).

b) Suppose $\phi \in \mathscr{S}$, show that

$$\Delta_i^h \phi \to D_i\phi, \qquad \text{as } h \to 0,$$

in $\mathscr{S}$, where Δ_i^h is the difference quotient defined in Example 2.

Exercise 2.1 (*)**.** Suppose that we work over $\mathbb{R}^n$ and that $f, g, h \in \mathscr{S}$.

a) Show that for any multi-index α, we have that $D^\alpha f \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, i.e. that

$$||D^\alpha f||_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.$$

b) Define

$$\begin{aligned} F \quad : \quad & \mathbb{R}^n \times \mathbb{R}^n, \\ & (x, y) \mapsto f(x)g(y - x). \end{aligned}$$

Show that $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

c) For each $x \in \mathbb{R}^n$, set

$$\begin{aligned} G_x \quad : \quad & \mathbb{R}^n \times \mathbb{R}^n, \\ & (y, z) \mapsto f(y)g(z)h(x - y - z). \end{aligned}$$

Show that $G_x \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

Exercise 2.2. Show that Theorem 1.10 holds under the alternative hypotheses:

a) $f \in L^1(\mathbb{R}^n)$, $g \in C^k(\mathbb{R}^n)$ with $\sup_{\mathbb{R}^n} |D^\alpha g| < \infty$ for all $|\alpha| \le k$.

b) $f \in L^1(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, $g \in C^k(\mathbb{R}^n)$.

Exercise 2.3. a) Prove the following identities for $r, s > 0$ and $x \in \mathbb{R}^n$:

i) $B_r(x) + B_s(0) = B_{r+s}(x)$

ii) $\overline{B_r(x)} + B_s(0) = B_{r+s}(x)$

iii) $\overline{B_r(x)} + \overline{B_s(0)} = \overline{B_{r+s}(x)}$

Suppose that $A, B \subset \mathbb{R}^n$. Show that:

b) If one of A or B is open, then so is $A + B$.

c) If A and B are both bounded, then so is $A + B$.

d) If A is closed and B is compact, then $A + B$ is closed.

e) If A and B are both compact, then so is $A + B$.

Exercise 2.4. Show that if $f \in C_0^k(\mathbb{R}^n)$ and $g \in C_0^l(\mathbb{R}^n)$ then $f \star g \in C_0^{k+l}(\mathbb{R}^n)$. Conclude that $\mathscr{D}(\mathbb{R}^n)$ is closed under convolution.

Exercise 2.5 (*)**.** Suppose that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive integrable simple function,

a) Show that Minkowski's integral inequality holds for the case $p = 1$:

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right| dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)|\, dydx$$

b) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$*, consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

c) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||f||_p = 1$ and $||g||_q = 1$ then

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq 1.$$

Deduce Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \ \leq ||f||_p\, ||g||_q\,, \qquad \text{for all } f \in L^p(\mathbb{R}^n), \quad g, \in L^q(\mathbb{R}^n).$$

d) Set $G(y) = \left(\int_{\mathbb{R}^n} F(x,y)dx\right)^{p-1}$

i) Show that if $q = \frac{p}{p-1}$:

$$||G||_{L^q(\mathbb{R}^n)} = \left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|^{p-1}_{L^p(\mathbb{R}^n)}$$

ii) Show that:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|^{p}_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G(y)F(x,y)dy \right) dx$$

iii) Applying Hölder's inequality, deduce:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|^{p}_{L^p(\mathbb{R}^n)} \leq ||G||_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} ||F(x,\cdot)||_{L^p(\mathbb{R}^n)}\, dx$$

e) Deduce that Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right|^p dy \right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |F(x,y)|^p\, dy \right]^{\frac{1}{p}} dx$$

holds for any measurable function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$, where $1 \leq p < \infty$.

Exercise 2.6. a) Show that δ_x, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

b) Show that T_f, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

c) By constructing a suitable sequence of smooth functions show that the order of $P.V. \left(\frac{1}{x}\right)$ is one.

Exercise 2.7. Suppose $f \in L^1_{loc.}(\Omega)$. Take ϕ_ϵ as in Theorem 1.13, and define for x with $d(x, \partial\Omega) \geq \epsilon$:

$$f_\epsilon(x) := T_f \left[\tau_x \check{\phi}_\epsilon\right].$$

Show that for any compact $K \subset \Omega$:

$$||f_\epsilon - f||_{L^1(K)} \to 0$$

as $\epsilon \to 0$.
[Hint: follow the proof of Theorem 2.2, but use part b) of Theorem 1.13]

Exercise 2.8. a) Show that if $f_1, f_2 \in C^0(\Omega)$ and $a \in C^\infty(\Omega)$, then

$$aT_{f_1} + T_{f_2} = T_{af_1+f_2}$$

b) Show that if $f \in C^k(\Omega)$ then

$$D^\alpha T_f = T_{D^\alpha f}$$

for $|\alpha| \leq k$. Deduce that $\iota \circ D^\alpha = D^\alpha \circ \iota$.

c) Deduce that if $f \in C^k(\Omega)$ then

$$\sum_{|\alpha| \leq k} a_\alpha D^\alpha T_f = T_{Lf}.$$

where

$$Lf = \sum_{|\alpha| \leq k} a_\alpha D^\alpha f$$

Exercise 2.9. a) Show that for $f, g \in C^0_0(\mathbb{R}^n)$:

$$T_{f \star g} = T_f \star T_g.$$

b) Show that convolution is linear in both of its arguments, i.e. if $u_i \in \mathscr{D}'(\mathbb{R}^n)$ and u_3, u_4 have compact support then

$$(u_1 + au_2) \star u_3 = u_1 \star u_3 + au_2 \star u_3$$

and

$$u_1 \star (u_3 + au_4) = u_1 \star u_3 + au_1 \star u_4$$

where $a \in \mathbb{C}$ is a constant.

Exercise 2.10. a) Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ then

$$\delta_0 \star \phi = \phi$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ has compact support, then

$$\delta_0 \star u = u$$

Exercise 3.1. a) Suppose that $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{E}(\Omega)$ is a sequence such that $\phi_j \to \phi$ in $\mathscr{E}(\Omega)$, and $\chi \in \mathscr{D}(\Omega)$. Show that

$$\chi\phi_j \to \chi\phi \quad \text{in } \mathscr{D}(\Omega).$$

b) Show that if $\psi \in \mathscr{E}(\Omega)$, then there exists a sequence $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{D}(\Omega)$ such that $\phi_j \to \psi$ in $\mathscr{E}(\Omega)$.
[Hint: Take an exhaustion of Ω by compact sets and apply Lemma 1.14]

Exercise 3.2. a) Prove Lemma 2.11.
[Hint: You should argue in a similar fashion to the proof of Lemma 2.9]

b) Show that we have the inclusions:

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}' \subset \mathscr{D}'(\mathbb{R}^n).$$

Exercise 3.3. Let $\{a_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence of real numbers. Define for $\phi \in C^{\infty}(\mathbb{R})$:

$$u[\phi] = \sum_{j=1}^{\infty} a_j \phi(j)$$

provided that the sum converges. Give necessary and sufficient conditions on a_j such that:

a) $u \in \mathscr{E}'(\mathbb{R})$.

b) $u \in \mathscr{S}'$.

c) $u \in \mathscr{D}'(\mathbb{R})$.

Exercise 3.4. For $\xi \in \mathbb{R}^n$, define $e_\xi(x) = e^{i\xi\cdot x}$. Show that $T_{e_\xi} \in \mathscr{S}'$, and that:

$$T_{e_\xi} \to 0, \qquad \text{as } |\xi| \to \infty$$

in the topology[1] of $\mathscr{S}'$.

Exercise 3.5. Calculate the Fourier transform of the following functions $f \in L^1(\mathbb{R})$:

a) $f(x) = \dfrac{\sin x}{1+x^2}$.

b) $f(x) = \dfrac{1}{\epsilon^2 + x^2}$, for $\epsilon > 0$ a constant.

c) $f(x) = \sqrt{\dfrac{\sigma}{t}} e^{-\sigma\frac{(x-y)^2}{t}}$, where $\sigma > 0$, $t > 0$ and y are constants.

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[1] This is defined precisely as the topology of $\mathscr{D}'(\Omega)$, *mutatis mutandis*.

*d) $f(x) = \dfrac{1}{\cosh x}$.

Exercise 3.6. Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$. Fix $\epsilon > 0$. Show that there exists $f_\epsilon \in C^1_0(\mathbb{R}^n)$ such that

$$||f - f_\epsilon||_{L^1(\mathbb{R}^n)} + ||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

[Hint: First construct, for large R, a smooth cut-off function $\chi_R(x)$ with $\chi_R(x) = 1$ for $|x| < R$, $\chi_R(x) = 0$ for $|x| > 2R$ and $|D\chi_R(x)| < C$, where C is independent of R.]

Exercise 3.7. Suppose $f \in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$.

a) Show that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi \in \mathbb{R}^n} \left| D^\alpha \hat{f}(\xi) \right| \leq R^{|\alpha|} \, ||f||_{L^1(\mathbb{R}^n)}$$

b) Show that $\hat{f}$ is real analytic, with an infinite radius of convergence, i.e.:

$$\hat{f}(\xi) = \sum_\alpha D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!}$$

holds for all $\xi \in \mathbb{R}^n$.

c) Show that if $\hat{f}(\xi)$ vanishes on an open set, it must vanish everywhere.
[Hint: use part i) of Lemma 3.2]

You may assume the following form of Taylor's theorem. Suppose $g \in C^{k+1}(\overline{B_r(0)})$. Then for $x \in B_r(0)$:

$$g(x) = \sum_{|\alpha| \leq k} D^\alpha g(0) \frac{\xi^\alpha}{\alpha!} + \sum_{|\beta| = k+1} R_\beta(x) x^\beta$$

where the remainder $R_\beta(x)$ satisfies the following estimate in $B_r(0)$:

$$|R_\beta(x)| \leq \frac{1}{\beta!} \max_{|\alpha| = |\beta|} \max_{y \in \overline{B_r(0)}} |D^\alpha g(y)| \, .$$

See §A.1 of the notes for notation.

Exercise 4.1. Consider the following ODE problem. Given $f : \mathbb{R} \to \mathbb{C}$, find ϕ such that:

$$-\phi'' + \phi = f. \tag{1}$$

a) Show that if $f \in \mathscr{S}$, there is a unique $\phi \in \mathscr{S}$ solving (1), and give an expression for $\hat{\phi}$.

b) Show that

$$\phi(x) = \int_{\mathbb{R}} f(y) G(x-y) dy$$

where

$$G(x) = \begin{cases} \frac{1}{2} e^x & x < 0, \\ \frac{1}{2} e^{-x} & x \geq 0. \end{cases}$$

Exercise 4.2. Suppose $f \in L^1(\mathbb{R}^3)$ is a radial function, i.e. $f(Rx) = f(x)$, whenever $R \in SO(3)$ is a rotation.

a) Show that $\hat{f}$ is radial.

b) Suppose that $\xi = (0, 0, \zeta)$. By writing the Fourier integral in polar coordinates, show that

$$\hat{f}(\xi) = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r) e^{-i\zeta r \cos\theta} r^2 \sin\theta d\theta dr d\phi.$$

c) Making the substitution $s = \cos\theta$, and using the fact that $\hat{f}$ is radial, deduce:

$$\hat{f}(\xi) = 4\pi \int_0^{\infty} f(r) \frac{\sin r|\xi|}{r|\xi|} r^2 dr$$

for any $\xi \in \mathbb{R}^n$.

Exercise 4.3. (*) Suppose that $f, g \in L^2(\mathbb{R}^n)$, and denote the Fourier-Plancherel transform by $\overline{\mathcal{F}}$. You may assume any results already established for the Fourier transform.

a) Show that

$$(f, g) = \frac{1}{(2\pi)^n} \left(\overline{\mathcal{F}}[g], \overline{\mathcal{F}}[g] \right).$$

b) Recall that $\check{f}(y) = f(-y)$. Show that:

$$\overline{\mathcal{F}} \left[\overline{\mathcal{F}}[f] \right] = (2\pi)^n \check{f}.$$

Hence, or otherwise, deduce that $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bijection, and that $\overline{\mathcal{F}}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear map.

c) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \lim_{R\to\infty} \int_{B_R(0)} f(x)e^{-ix\cdot\xi}dx$$

with convergence in the sense of $L^2(\mathbb{R}^n)$.

d) Suppose that $f \in C^1(\mathbb{R}^n)$ and $f, D_j f \in L^2(\mathbb{R}^n)$. Show that $\xi_j\overline{\mathcal{F}}[f](\xi) \in L^2(\mathbb{R}^n)$ and:

$$\overline{\mathcal{F}}[D_j f](\xi) = i\xi_j\overline{\mathcal{F}}[f](\xi)$$

e) For $x \in \mathbb{R}$ let:

$$f(x) = \frac{\sin x}{x}$$

i) Show that $f \in L^2(\mathbb{R})$.

ii) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \begin{cases} \pi & -1 < \xi < 1, \\ 0 & |\xi| \geq 1. \end{cases}$$

f) i) Show that for all $x \in \mathbb{R}^n$:

$$|f \star g(x)| \leq ||f||_{L^2(\mathbb{R}^n)} \, ||g||_{L^2(\mathbb{R}^n)} \, .$$

ii) Show that $f \star g \in C^0(\mathbb{R}^n)$ and:

$$f \star g = \mathcal{F}^{-1}\left[\overline{\mathcal{F}}[f] \cdot \overline{\mathcal{F}}[g]\right]$$

where:

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}d\xi.$$

[Hint for parts a), b), d), f): approximate by Schwartz functions]

Exercise 4.4. Work in $\mathbb{R}^3$. For $k > 0$, define the function:

$$G(x) = \frac{e^{-k|x|}}{4\pi\,|x|}$$

a) Show that $G \in L^1(\mathbb{R}^3)$.

b) Show that:

$$\hat{G}(\xi) = \frac{1}{|\xi|^2 + k^2}$$

[Hint: use Exercise 4.2, part c)]

Exercise 4.5. Consider the inhomogeneous Helmholtz equation on $\mathbb{R}^3$:

$$-\Delta\phi + k^2\phi = f \tag{2}$$

where $f \in \mathscr{S}$. Show that there exists a unique $\phi \in \mathscr{S}$ satisfying (2) given by:

$$\phi(x) = \int_{\mathbb{R}^3} f(y)G(x-y)dy,$$

where

$$G(x) = \frac{e^{-k|x|}}{4\pi\,|x|}.$$

[Hint: first derive an equation satisfied by $\hat{\phi}$]

Exercise 4.6. Verify that if $f \in L^1_{loc.}$ is such that $T_f \in \mathscr{S}'$, then:

$$\tau_x T_f = T_{\tau_x f}, \quad \text{and} \quad \check{T}_f = T_{\check{f}}$$

Exercise 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and define $f_R(x) = f(x)\mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$.

b) Show that:

$$T_{f_R} \to T_f \text{ in } \mathscr{S}' \text{ as } R \to \infty.$$

c) Show that:

$$\hat{f}_R(\xi) = 2i\frac{\cos R\xi - 1}{\xi}$$

d) For $\phi \in \mathscr{S}$, show that:

$$T_{\hat{f}_R}[\phi] = -2i\int_0^\infty \frac{\phi(x) - \phi(-x)}{x}dx + 2i\int_0^\infty \left(\frac{\phi(x) - \phi(-x)}{x}\right)\cos Rx dx$$

e) By applying the Riemann-Lebesgue Lemma, or otherwise, show that for any $\psi \in \mathscr{S}$:

$$\int_0^\infty \psi(x)\cos Rx dx \to 0$$

as $R \to \infty$.

f) Deduce that

$$\widehat{T_f} = -2iP.V.\left(\frac{1}{x}\right)$$

g) Write down $\widehat{T_H}$, where H is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Exercise 5.1. Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ and let:

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v.$$

Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset K$ for some compact $K \subset \mathbb{R}^n$ then

$$u[\phi] = \sum_{g \in A} \tau_g v[\phi],$$

for some finite set $A \subset \mathbb{Z}^n$ which depends only on K. Deduce that u defines a distribution.

Exercise 5.2. Recall that for $x \in \mathbb{R}^n$:

$$||x||_1 := \sum_{i=1}^{n} |x_i| .$$

For $k \in \mathbb{N}$ set:

$$Q_k = \left\{ g \in \mathbb{Z}^n : k - \frac{1}{2} \leq ||g||_1 < k + \frac{1}{2} \right\}$$

a) Show that:

$$\#Q_k = (2k+1)^n - (2k-1)^n$$

so that $\#Q_k \leq c(1+k)^{n-1}$ for some $c > 0$.

b) By applying the Cauchy-Schwartz identity to estimate $a \cdot b$ for $a = (1, \ldots, 1)$ and $b = (|g_1|, \ldots, |g_n|)$, deduce that:

$$||g||_1 \leq \sqrt{n}\, |g|$$

c) Show that there exists a constant $C > 0$, depending only on n such that:

$$\sum_{g \in \mathbb{Z}^n; ||g||_1 \leq K} \frac{1}{(1+|g|)^{n+1}} \leq 1 + C \sum_{k=1}^{K} \frac{1}{k^2}$$

holds for all $K \in \mathbb{N}$. Deduce that:

$$\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty.$$

Exercise 5.3. Show that if c_g satisfy:

$$|c_g| \leq K(1+|g|)^N$$

for some $K > 0$ and $N \in \mathbb{N}$, then:

$$\sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}$$

converges in $\mathscr{S}'$.

Exercise 5.4. Suppose $f \in L^p_{loc.}(\mathbb{R}^n)$ is a periodic function. Fix $\epsilon > 0$, and let:

$$Q = \{x \in \mathbb{R}^n : |x_j| < 1, j = 1, \ldots, n\}, \qquad q = \left\{x \in \mathbb{R}^n : |x_j| < \frac{1}{2}, j = 1, \ldots, n\right\}$$

a) Show that there exists $h_\epsilon \in C^\infty(\mathbb{R}^n)$ with:

$$\operatorname{supp} h_\epsilon \subset Q$$

such that:

$$||f\mathbb{1}_q - h_\epsilon||_{L^p(\mathbb{R}^n)} < \epsilon.$$

Define

$$f_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_g h_\epsilon$$

b) Show that f_ϵ is smooth and periodic.

c) Show that there exists a constant c_n depending only on n such that:

$$||f - f_\epsilon||_{L^p(q)} < c_n \epsilon.$$

Exercise 5.5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = x \ \text{ for } \ |x| < \frac{1}{2}, \qquad f(x+1) = f(x).$$

Show that:

$$f(x) = \sum_{n \in \mathbb{Z}, n \neq 0} \frac{i(-1)^n}{2\pi n} e^{2\pi i n x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n x),$$

with convergence in $L^2_{loc.}(\mathbb{R})$.

Exercise 5.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} -1 & -\frac{1}{2} < x \leq 0 \\ 1 & 0 < x \leq \frac{1}{2} \end{cases}, \qquad f(x+1) = f(x).$$

a) Show that:

$$f(x) = \frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{2}{2n+1} e^{2\pi i (2n+1) x} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[2\pi(2n+1)x\right]$$

With convergence in $L^2_{loc.}(\mathbb{R}^n)$.

Define the partial sum:

$$S_N(x) = 8 \sum_{n=0}^{N-1} \frac{1}{2\pi(2n+1)} \sin\left[2\pi(2n+1)x\right].$$

b) Show that:

$$S_N(x) = 8\int_0^x \sum_{n=0}^{N-1} \cos\left[2\pi(2n+1)t\right] dt.$$

c) Show that:

$$\cos\left[2\pi(2n+1)t\right]\sin 2\pi t = \frac{1}{2}\left(\sin\left[2\pi(2n+2)t\right] - \sin\left[4\pi n t\right]\right)$$

And deduce:

$$S_N(x) = 8\int_0^x \frac{\sin 4\pi N t}{2\sin 2\pi t} dt.$$

d) Show that the first local maximum of S_N occurs at $x = \frac{1}{4N}$, and:

$$S_N\left(\frac{1}{4N}\right) \geq 8\int_0^{\frac{1}{4N}} \frac{\sin 4\pi N t}{4\pi t} dt = \frac{2}{\pi}\int_0^{\pi} \frac{\sin s}{s} ds \simeq 1.179\ldots$$

e) Conclude that the sum in part *a*) does not converge uniformly.

This lack of uniform convergence of a Fourier series at a point of discontinuity is known as Gibbs Phenomenon.

Exercise 5.7. (*) Suppose that $\lambda = \{\lambda_1, \ldots \lambda_n\}$ is a basis for $\mathbb{R}^n$. We define the lattice generated by λ to be:

$$\Lambda = \left\{ \sum_{j=1}^{n} z_j \lambda_j : z_j \in \mathbb{Z} \right\}.$$

Define the fundamental cell:

$$q_\Lambda = \left\{ \sum_{j=1}^{n} x_j \lambda_j : |x_j| < \frac{1}{2} \right\}.$$

We say that $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic if:

$$\tau_g u = u \qquad \text{for all } g \in \Lambda.$$

a) Show that there exists $\psi \in C_0^\infty(2q_\Lambda)$ such that $\psi \geq 0$ and

$$\sum_{g\in\Lambda} \tau_g \psi = 1.$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic and ψ, ψ' are both as in part *a*), then

$$\frac{1}{|q_\Lambda|} u[\psi] = \frac{1}{|q_\Lambda|} u[\psi'] =: M(u)$$

c) Define the *dual lattice* by:

$$\Lambda^* := \{x \in \mathbb{R}^n : g \cdot x \in 2\pi\mathbb{Z}, \ \forall g \in \Lambda\}$$

Show that there exists a basis $\lambda^* = \{\lambda_1^*, \dots \lambda_n^*\}$ such that $\lambda_j^* \cdot \lambda_k = \delta_{jk}$, and Λ^* is the lattice induced by λ^*.

d) Show that if $g \in \Lambda^*$ then e_g is $\Lambda-$periodic.

e) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$\hat{u} = \sum_{g \in \Lambda^*} c_g \delta_g$$

for some $c_g \in \mathbb{C}$ satisfying $|c_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$.

f) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$u = \sum_{g \in \Lambda^*} d_g T_{e_g}$$

where $|d_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$ are given by:

$$d_g = M(e_{-g} u)$$

Exercise 5.8. Suppose $s \geq 0$.

a) Show that $\mathscr{S} \subset H^s(\mathbb{R}^n)$.

b) Suppose $f \in H^s(\mathbb{R}^n)$. Show that given $\epsilon > 0$ there exists $f_\epsilon \in \mathscr{S}$ with:

$$||f - f_\epsilon||_{H^s(\mathbb{R}^n)} < \epsilon.$$

Hint: First find $g_\epsilon \in \mathscr{S}$ such that

$$\left|\left|(\hat{f} - g_\epsilon)(1 + |\xi|)^s\right|\right|_{L^2(\mathbb{R}^n)} < \epsilon.$$

c) Show that

$$||f||_{H^s(\mathbb{R}^n)} \leq ||f||_{H^t(\mathbb{R}^n)}$$

for $t \geq s$. Deduce that:

$$||f||_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} ||f||_{H^s(\mathbb{R}^n)}$$

Hint: Use Parseval's formula

d) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

Exercise 5.9. Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $u(t,x)$ is the solution of the heat equation with initial data u_0. Explicitly, u is given by:

$$u(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi \cdot x} d\xi,$$

for $t > 0$.

a) Show that:

$$||u(t,\cdot)||_{L^2(\mathbb{R}^n)} \leq ||u_0||_{L^2(\mathbb{R}^n)},$$

b) Show that:

$$u(t,x) = u_0 \star K_t(x)$$

where the *heat kernel* is given by:

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

c) Suppose that $u_0 \geq 0$. Show that $u \geq 0$, and:

$$||u(t,\cdot)||_{L^1(\mathbb{R}^n)} = ||u_0||_{L^1(\mathbb{R}^n)}.$$

[Hint: Lemma 1.9 may be useful]

Exercise 5.10. Consider the Schrödinger equation:

$$\left\{ \begin{array}{rcl} u_t & = & i\Delta u \qquad \text{in } (0,T) \times \mathbb{R}^n, \\ u & = & u_0 \qquad \text{on } \{0\} \times \mathbb{R}^n \end{array} \right. \tag{3}$$

Suppose $u_0 \in H^2(\mathbb{R}^n)$.

a) Show that (3) admits a unique solution u such that

$$\boldsymbol{u} \in C^0([0,T); H^2(\mathbb{R}^n)) \cap C^1((0,T); L^2(\mathbb{R}^n)),$$

whose spatial Fourier-Plancherel transform is given by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi) e^{-it|\xi|^2}.$$

b) Show that:

$$||u(t,\cdot)||_{H^2(\mathbb{R}^n)} = ||u_0||_{H^2(\mathbb{R}^n)}$$

*c) For $t > 0$, let $K_t \in L^1_{loc.}(\mathbb{R}^n)$ be given by:

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}},$$

where for n odd we take the usual branch cut so that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. For $\epsilon > 0$ set $K_t^\epsilon(x) = e^{-\epsilon|x|^2} K_t(x)$.

i) Show that $T_{K_t^\epsilon} \to T_{K_t}$ in $\mathscr{S}'$ as $\epsilon \to 0$.

ii) Show that if $\Re(\sigma) > 0$, then:

$$\int_{\mathbb{R}} e^{-\sigma x^2 - ix\xi} dx = \sqrt{\frac{\pi}{\sigma}} e^{-\frac{\xi^2}{4\sigma}}.$$

iii) Deduce that

$$\widehat{K_t^\epsilon}(\xi) = \left(\frac{1}{1+4it\epsilon}\right)^{\frac{n}{2}} e^{\frac{-it|\xi|^2}{1+4it\epsilon}}$$

iv) Conclude that:

$$\widehat{T_{K_t}} = T_{\tilde{K}_t},$$

where $\tilde{K}_t = e^{-it|\xi|^2}$.

*d) Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$. Show that for $t > 0$:

$$u(t,x) = \int_{\mathbb{R}^n} u_0(y) K_t(x-y) dy,$$

and deduce:

$$\sup_{t>0, x\in\mathbb{R}^n} |u(t,x)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} \left|\left|\hat{u}_0\right|\right|_{L^1(\mathbb{R}^n)}.$$

This type of estimate which shows us that (locally) solutions to the Schrödinger equation decay in time is known as a *dispersive estimate.*

Exercise 5.11. Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := (-T,T) \times \mathbb{R}^3_*$ and $|x| = r$. You may assume the result that if $u = u(r,t)$ is radial, we have

$$\Delta u(|x|, t) = \Delta u(r,t) = \frac{\partial^2 u}{\partial r^2}(r,t) + \frac{2}{r}\frac{\partial u}{\partial r}(r,t)$$

a) Suppose $u(x,t) = \frac{1}{r} v(r,t)$ for some function v. Show that u solves the wave equation on $\mathbb{R}^3_* \times (0,T)$ if and only if v satisfies the one-dimensional wave equation

$$-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} = 0$$

on $(0,\infty) \times (-T,T)$.

b) Suppose $f, g \in C^2_c(\mathbb{R})$. Deduce that

$$u(x,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

c) Show that if $f \in C^3_c(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all s) then

$$u(x,t) = \frac{f(r+t) + f(r-t)}{2r}$$

extends as a C^2 function which solves the wave equation on $S_T := (-T,T) \times \mathbb{R}^3$, with

$$u(0,t) = f'(t).$$

*d) By considering a suitable sequence of functions f, or otherwise, deduce that there exists no constant C independent of u such that the estimate

$$\sup_{S_T} (|u| + |u_t|) \leq C \sup_{\Sigma_0} (|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.

Exercise A.1. Suppose that $\lambda_1 \lambda_2 \geq 0$ and that $U \subset X$ is a convex subset of a vector space X. Show that:

$$\lambda_1 U + \lambda_2 U = (\lambda_1 + \lambda_2) U.$$

Exercise A.2. a) Suppose that (S, τ) is a topological space, and that β is a base for τ. Show that:

i) If $x \in S$, then there exists some $B \in \beta$ with $x \in B$.

ii) If B_1, $B_2 \in \beta$, then for every $x \in B_1 \cap B_2$ there exists $B \in \beta$ with:

$$x \in B \qquad B \subset B_1 \cap B_2.$$

b) Conversely, suppose that one is given a set S and a collection β of subsets of S satisfying *i*), *ii*) above. Define τ by:

$$U \in \tau \iff \text{for all } x \in U, \text{ there exists } B \in \beta \text{ such that } x \in B \text{ and } B \subset U.$$

i.e. τ is the set of all unions of elements of β. Show that (S, τ) is a topological space, with base β. We say that τ is the topology generated by β

c) Suppose that β, β' both satisfy conditions *i*), *ii*) above and generate topologies τ, τ' respectively. Moreover, suppose that if $B \in \beta$ then for every $x \in B$ there exists $B' \in \beta'$ satisfying

$$x \in B', \qquad \text{and} \qquad B' \subset B$$

Then $\tau \subset \tau'$.

Exercise A.3. Suppose (S_1, τ_1), (S_2, τ_2) and (S_3, τ_3) are topological spaces, and that $f : S_1 \times S_2 \to S_3$ is a continuous map. Show that for each $a \in S_1$ and $b \in S_2$, the maps

$$f_a \ : \ \begin{array}{l} S_2 \to S_3, \\ y \mapsto f(a, y), \end{array} \qquad\qquad f^b \ : \ \begin{array}{l} S_1 \to S_3, \\ x \mapsto f(x, b), \end{array}$$

are continuous.

The condition that f is continuous with respect to the product topology is sometimes called *joint continuity*, while the continuity of f_a, f^b is called *separate continuity.* Thus joint continuity implies separate continuity. The converse is not true.

Exercise A.4. Show that the base

$$\beta_{\mathbb{Q}} = \{(p, q) : p, q \in \mathbb{Q}, \ p < q\},$$

generates the standard topology on $\mathbb{R}$.

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Exercise A.5. Suppose that (S, d) is a metric space. Show that S is Hausdorff with respect to the metric topology.

Exercise A.6. Let us take $X = \mathbb{R}^n$, thought of as a vector space over $\mathbb{R}$ and define:

$$||(x_1, \ldots, x_n)||_p = (|x_1|^p + \ldots |x_n|^p)^{\frac{1}{p}}, \qquad p \geq 1.$$

a) Show that $(\mathbb{R}^n, ||\cdot||_p)$ is a normed vector space:

i) First check that the positivity and homogeneity property are satisfied.

ii) Establish the triangle inequality for the special case $p = 1$.

iii) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, *consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

iv) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||x||_p = 1$ and $||y||_q = 1$ then

$$\sum_{i=1}^{n} |x_i y_i| \leq 1.$$

Deduce Hölder's inequality:

$$\sum_{i=1}^{n} |x_i y_i| \ \leq ||x||_p \, ||y||_q, \qquad \text{for all } x, y, \in \mathbb{R}^n.$$

v) Show that

$$||x+y||_p^p \leq \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

vi) Apply Hölder's inequality to deduce:

$$||x+y||_p^p \leq \left(||x||_p + ||y||_p\right) ||x+y||_p^{p-1}$$

and conclude

$$||x+y||_p \leq ||x||_p + ||y||_p.$$

b) Show that the metric topology of $(\mathbb{R}^n, ||\cdot||_p)$ agrees with the standard topology.
Hint: Use part c) of Exercise A.2

Exercise A.7 $(\star)$**.** Let $X = C[0, 1]$, the set of continuous functions on the closed interval $[0, 1]$. For $f \in X$, $p \geq 0$ define:

$$||f||_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

a) Show that X is a vector space over $\mathbb{R}$, where scalar multiplication and vector addition are defined pointwise.

b) Establish Hölder's inequality:

$$||fg||_1 \leq ||f||_p \, ||g||_q$$

for $p, q > 1$ with $p^{-1} + q^{-1} = 1$.

c) Show that $(X, ||\cdot||_p)$ is a normed space.

d) Suppose $p \leq p'$. Show that:

$$||f||_p \leq ||f||_{p'}$$

e) Let τ_p be the metric topology of $(X, ||\cdot||_p)$. Show that if $p \leq p'$:

$$\tau_p \subset \tau_{p'}.$$

f) Consider the sequence of functions:

$$f_n(x) = \begin{cases} n^{\gamma-1} & 0 \leq x < \frac{1}{n} \\ \frac{1}{n} x^{-\gamma} & \frac{1}{n} \leq x \leq 1 \end{cases}$$

where $n = 1, 2, \ldots$

i) Show that $f_n \in C[0,1]$ and

$$\lim_{n\to\infty} ||f_n||_p = \begin{cases} 0 & \gamma < \frac{p+1}{p} \\ \left(\frac{p+1}{p}\right)^{\frac{1}{p}} & \gamma = \frac{p+1}{p} \\ \infty & \gamma > \frac{p+1}{p} \end{cases}$$

ii) By choosing γ carefully, show that if $p < p'$ then

$$\tau_{p'} \not\subset \tau_p.$$

Hint: in parts b), c) follow the same steps as for the finite dimensional case in Exercise A.6.

Exercise A.8. Verify that if (D, s) is a metric space, then the metric topology defines the same notions of convergence and continuity as the standard definitions for a metric space.

Exercise A.9. Let (X, τ) be a topological vector space

a) Show that if $(x_n)_{n=1}^{\infty}$ is a τ-Cauchy sequence, then $\{x_n\}_{n=1}^{\infty}$ is bounded.

b) Fix a local base $\dot{\beta}$. Show that a sequence $(x_n)_{n=1}^{\infty}$ is τ-Cauchy if and only if for any $B \in \dot{\beta}$ we can find an integer N such that

$$x_n - x_m \in B, \qquad \text{for all } n, m \geq N.$$