Exercise 1.1. Consider the meromorphic function

$$F(z) = \frac{1}{1+z^2} \frac{\cos\left[(\pi-\theta)z\right]}{2\sin\pi z}$$

a) Show that the poles of $F(z)$ occur at $z = \pm i$ and $z = n$, for $n \in \mathbb{Z}$, with:

$$\mathrm{Res}[F, \pm i] = -\frac{\cosh(\pi-\theta)}{4\sinh\pi}, \qquad \mathrm{Res}[F, n] = \frac{\cos(n\theta)}{2\pi(1+n^2)}.$$

Solution As a ratio of holomorphic functions, F is clearly meromorphic. Poles occur when the function $(1+z^2)\sin\pi z$ has a zero. This can occur when either $1+z^2=0$, which happens if and only if $z = \pm i$ or alternatively when $\sin\pi z = 0$, which occurs precisely when $z \in \mathbb{Z}$. We have

$$F(z) = \frac{1}{1+z^2} \frac{\cos\left[(\pi-\theta)z\right]}{2\sin\pi z} = \frac{1}{z\mp i}\frac{1}{z\pm i}\frac{\cos\left[(\pi-\theta)z\right]}{2\sin\pi z}$$

so that

$$F(z) = \frac{1}{z\mp i} G_\pm(z)$$

where

$$G_\pm(z) = \frac{1}{z\pm i}\frac{\cos\left[(\pi-\theta)z\right]}{2\sin\pi z}$$

is holomorphic at $z = \pm i$. Thus

$$\mathrm{Res}[F, \pm i] = G_\pm(\pm i) = \frac{1}{\pm 2i}\frac{\cos\left[(\pi-\theta)i\right]}{2\sin\pm(\pi i)} = -\frac{\cosh(\pi-\theta)}{4\sinh\pi}.$$

b) For $N \in \mathbb{N}$, let $\Gamma_i^{(N)}$ be the curves

$$\Gamma_1^{(N)}(t) = \left(N+\frac{1}{2}\right)(1+it), \qquad -1 \le t \le 1$$

$$\Gamma_2^{(N)}(t) = \left(N+\frac{1}{2}\right)(-t+i), \qquad -1 \le t \le 1$$

$$\Gamma_3^{(N)}(t) = \left(N+\frac{1}{2}\right)(-1-it), \qquad -1 \le t \le 1$$

$$\Gamma_4^{(N)}(t) = \left(N+\frac{1}{2}\right)(t-i), \qquad -1 \le t \le 1$$

and let $\Gamma^{(N)}$ be the closed contour which results from following Γ_1, Γ_2, Γ_3 and Γ_4 in turn. Sketch $\Gamma^{(N)}$, together with the locations of the poles of F in the complex plane.

Solution The contour is a square centred at the origin of side length $2N+1$, traversed in an anti-clockwise direction. There are poles at $\pm i$ and along the real axis at integer points. The contour crosses the real axis exactly half-way between a pair of poles.

c) Show that if $0 \leq \theta \leq 2\pi$, then:

$$\left| \int_{\Gamma^{(N)}} F(z) dz \right| \leq \frac{C}{N},$$

where C is a fixed constant, independent of N.

Solution Let us write $w = x + iy$ with $x, y \in \mathbb{R}$. Then we have:

$$\begin{aligned} \sin w &= \sin x \cosh y + i \cos x \sinh y \\ \cos w &= \cos x \cosh y - i \sin x \sinh y \end{aligned}$$

Since $\sinh^2 y \leq \cosh^2 y$, we estimate:

$$\begin{aligned} |\cos w|^2 &= \cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y \\ &\leq \cos^2 x \cosh^2 y + \sin^2 x \cosh^2 y \\ &= \cosh^2 y \end{aligned}$$

so that $|\cos w| \leq \cosh(\Im w)$. Similarly, we have

$$\begin{aligned} |\sin w|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &\geq \cos^2 x \sinh^2 y + \sin^2 x \cosh^2 y \\ &= \sinh^2 y \end{aligned}$$

so $|\sin w| \geq |\sinh(\Re w)|$. Finally, note that if $\Re w = \pi\left(n + \frac{1}{2}\right)$, $n \in \mathbb{Z}$, we have:

$$\left| \sin\left[\pi \left(N + \frac{1}{2} \right) + iy \right] \right| = \cosh y$$

Let us take each component of the contour in turn. On $\Gamma_1^{(N)}$, we have $\Re(z) = N + \frac{1}{2}$, so that with the estimates above we have:

$$\left| \frac{\cos[(\pi - \theta)z]}{\sin \pi z} \right| \leq \frac{\cosh[(\pi - \theta)\Im z]}{\cosh[\pi \Im z]} \leq 1$$

where in the last inequality we have used $|(\pi - \theta)\Im z| \leq |\pi \Im z|$ and properties of cosh. An identical argument shows that the same bound holds on $\Gamma_3^{(N)}$. Now consider $\Gamma_2^{(N)}$ and $\Gamma_4^{(N)}$. We have:

$$\left| \frac{\cos[(\pi - \theta)z]}{\sin \pi z} \right| \leq \frac{\cosh\left[(\pi - \theta)\left(N + \frac{1}{2}\right)\right]}{\left|\sinh\left[\pi\left(N + \frac{1}{2}\right)\right]\right|}.$$

Now, recalling that $\sinh x = \cosh x - e^{-x}$, we see that as $N \to \infty$, we have:

$$\lim_{N\to\infty} \frac{\cosh\left[(\pi-\theta)\left(N+\frac{1}{2}\right)\right]}{\left|\sinh\left[\pi\left(N+\frac{1}{2}\right)\right]\right|} = \begin{cases} 0 & 0<\theta<2\pi \\ 1 & \theta = 0, 2\pi. \end{cases}$$

Since an infinite sequence which tends to a finite limit must be bounded, we have that

$$\frac{\cosh\left[(\pi-\theta)\left(N+\frac{1}{2}\right)\right]}{\left|\sinh\left[\pi\left(N+\frac{1}{2}\right)\right]\right|} \leq K$$

for some constant K, independent of N.

To summarise our calculations so far, we have shown that on $\Gamma^{(N)}$ we have:

$$\left|\frac{\cos\left[(\pi-\theta)z\right]}{\sin \pi z}\right| \leq K+1.$$

Now, note that $\left|1+z^2\right| = |z+i|\,|z-i|$ and that for $z \in \Gamma^{(N)}$ we have $|z \pm i| \geq N - \frac{1}{2} \geq \frac{1}{2}N$. Thus

$$\left|\frac{1}{1+z^2}\right| \leq \frac{4}{N^2},$$

and we conclude

$$\sup_{z\in\Gamma^{(n)}} = |F(z)| \leq \frac{2(K+1)}{N^2}.$$

Finally, noting that the length of $\Gamma^{(N)}$ is given by

$$\left|\Gamma^{(N)}\right| = 4(2N+1) \leq 12N,$$

we conclude that

$$\left|\int_{\Gamma^{(N)}} F(z)dz\right| \leq \sup_{z\in\Gamma^{(n)}} |F(z)| \times \left|\Gamma^{(N)}\right| \leq \frac{24(K+1)}{N}.$$

d) By applying Cauchy's residue theorem, show that for $0 \leq \theta \leq 2\pi$:

$$\left|\frac{\cosh(\pi-\theta)}{2\sinh\pi} - \sum_{n=-N}^{N} \frac{e^{in\theta}}{1+n^2}\right| \leq \frac{C}{N},$$

and conclude that

$$\frac{\cosh(\pi-\theta)}{2\sinh\pi} = \sum_{n=-\infty}^{\infty} \frac{e^{in\theta}}{1+n^2},$$

with the sum converging uniformly in θ.

Solution We have, by the residue theorem:

$$\frac{1}{2\pi i}\int_{\Gamma^{(N)}} F(z)dz = -\frac{\cosh(\pi-\theta)}{2\sinh\pi} + \sum_{n=-N}^{N} \frac{e^{in\theta}}{1+n^2}.$$

Taking the modulus and estimating the left hand side by the previous part we're done.

Exercise 1.2. Suppose that $f \in C^0(\mathbb{R})$ is a continuous function with period 2π, i.e. $f(\theta) = f(\theta + 2\pi)$. For $\theta \in [0, 2\pi)$, define:

$$\psi(\theta) := \int_0^\theta f(\alpha)\frac{\cosh(\pi - \theta + \alpha)}{2\sinh\pi}d\alpha + \int_\theta^{2\pi} f(\alpha)\frac{\cosh(-\pi - \theta + \alpha)}{2\sinh\pi}d\alpha$$

and extend ψ to a function on $\mathbb{R}$ by periodicity: $\psi(\theta) = \psi(\theta + 2\pi)$.

a) Show that $\psi \in C^0(\mathbb{R})$.

Solution Since an integral of a continuous function is continuous, the expression is manifestly continuous on the intercal $(0, 2\pi)$. To show $\psi \in C^0(\mathbb{R})$ we need to verify continuity, but this is straightforward as:

$$\psi(0) = \int_0^{2\pi} f(\alpha)\frac{\cosh(-\pi + \alpha)}{2\sinh\pi}d\alpha = \psi(2\pi)$$

b) By directly differentiating the formula, show that:

$$\psi'(\theta) = -\int_0^\theta f(\alpha)\frac{\sinh(\pi - \theta + \alpha)}{2\sinh\pi}d\alpha - \int_\theta^{2\pi} f(\alpha)\frac{\sinh(-\pi - \theta + \alpha)}{2\sinh\pi}d\alpha$$

and show that $\psi \in C^1(\mathbb{R})$.

Solution Note that if $f \in C^1$, then

$$F(x) = \int_0^x f(x, t)dt$$

then

$$F'(x) = f(x, x) + \int_0^x f_x(x, t)dt.$$

Thus for $\theta \in (0, 2\pi)$:

$$\begin{aligned}\psi'(\theta) = f(\theta)\frac{\cosh(\pi)}{2\sinh\pi} &- \int_0^\theta f(\alpha)\frac{\sinh(\pi - \theta + \alpha)}{2\sinh\pi}d\alpha \\ - f(\theta)\frac{\cosh(\pi)}{2\sinh\pi} &- \int_\theta^{2\pi} f(\alpha)\frac{\sinh(-\pi - \theta + \alpha)}{2\sinh\pi}d\alpha\end{aligned}$$

The first term on each line cancels and the result follows. The formula for $\psi'(\theta)$ immediately shows that $\psi' \in C^0(0, 2\pi)$. Again, all that remains to check to see that $\psi \in C^1(\mathbb{R})$ is the periodicity, which follows from:

$$\psi'(0) = -\int_0^{2\pi} f(\alpha)\frac{\sinh(-\pi + \alpha)}{2\sinh\pi}d\alpha = \psi'(2\pi).$$

c) Differentiating again, show that

$$\psi''(\theta) = -f(\theta) + \psi(\theta)$$

Conclude that $\psi \in C^2(\mathbb{R})$ is a solution to

$$-\psi''(\theta) + \psi(\theta) = f(\theta), \qquad \theta \in \mathbb{R}.$$

Solution We again differentiate to obtain for $\theta \in (0, 2\pi)$:

$$\begin{aligned}\psi''(\theta) = -f(\theta)\frac{\sinh(\pi)}{2\sinh\pi} + \int_0^\theta f(\alpha)\frac{\cosh(\pi-\theta+\alpha)}{2\sinh\pi}d\alpha \\ + f(\theta)\frac{\sinh(-\pi)}{2\sinh\pi} + \int_\theta^{2\pi} f(\alpha)\frac{\cosh(-\pi-\theta+\alpha)}{2\sinh\pi}d\alpha \\ = f(\theta) + \psi(\theta)\end{aligned}$$

Since $\psi'' = -f + \psi$, ψ'' is equal to a 2π-periodic continuous function, and clearly solves the equation.

Exercise 1.3. Let $\phi \in C^2(\mathbb{R})$ be 2π-periodic, i.e. $\phi(\theta) = \phi(\theta + 2\pi)$, and suppose that ϕ satisfies:

$$-\phi''(\theta) + \phi(\theta) = 0, \qquad \theta \in \mathbb{R}.$$

a) Show that if ϕ attains a maximum at $\theta_0 \in \mathbb{R}$, then $\phi(\theta_0) \le 0$.

Solution At a maximum point $\phi''(\theta_0) \le 0$, so since equation holds $\psi(\theta_0) = \psi''(\theta_0) \le 0$.

b) Show that if ϕ attains a minimum at $\theta_0 \in \mathbb{R}$, then $\phi(\theta_0) \ge 0$.

Solution At a minimum point $\phi''(\theta_0) \ge 0$, so since equation holds $\psi(\theta_0) = \psi''(\theta_0) \ge 0$.

c) Show that $\phi \equiv 0$.

Solution Since ϕ is periodic and continuous it achieves its maximum and minimum on the interval $[0, 2\pi]$. However we must have $\phi \le 0$ at the maximum and $\phi \ge 0$ at the minumum, which implies that $\phi \equiv 0$.

d) Conclude that there is at most one $\psi \in C^2(\mathbb{R})$ satisfying $\psi(\theta) = \psi(\theta + 2\pi)$ and solving:

$$-\psi''(\theta) + \psi(\theta) = f(\theta), \qquad \theta \in \mathbb{R},$$

where $f \in C^0(\mathbb{R})$ is a given 2π-periodic function.

Solution Suppose ψ_1, ψ_2 are two solutions. Then $\phi = \psi_1 - \psi_2$ satisfies the conditions of the first part, and so must vanish.

Exercise 1.4. For $t \in \mathbb{R}$ let:

$$\chi(t) = \begin{cases} 0 & t \le 0 \\ e^{-\frac{1}{t}} & t > 0 \end{cases}$$

a) Show that $\chi \in C^\infty(\mathbb{R})$.

Solution Clearly χ is smooth on $t < 0$ and $t > 0$, so the only thing that needs to be shown is that all derivatives are continuous across $t = 0$. First, I claim that for $t > 0$:

$$\chi^{(n)}(t) = P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t}},$$

where $P_n(x)$ is a polynomial. Clearly this is true for $n = 0$. Suppose true for n, then:

$$\begin{aligned} \chi^{(n+1)}(t) &= \frac{d}{dt}\left[P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t}}\right] \\ &= -\frac{1}{t^2}P_n'\left(\frac{1}{t}\right) e^{-\frac{1}{t}} + \frac{1}{t^2}P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t}} \\ &= P_{n+1}\left(\frac{1}{t}\right) e^{-\frac{1}{t}}, \end{aligned}$$

where:

$$P_{n+1}(s) = s^2\left[P_n(s) - P_n'(s)\right]$$

is a polynomial, since $P_n(s)$ is by assumption. Thus by induction:

$$\chi^{(n)}(t) = P_n\left(\frac{1}{t}\right) e^{-\frac{1}{t}},$$

holds for all n. Now, since 'an exponential always beats a power', we deduce that

$$\lim_{t \searrow 0} \chi^{(n)}(t) = 0.$$

Since $\chi^{(n)}(t) = 0$ for all $t < 0$, we have that $\chi^{(n)}$ is continuous across $t = 0$ for all n and we are done.

b) Show that there exists a function $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

i) $0 \le \psi \le 1$

ii) $\operatorname{supp} \psi \subset B_2(0)$

iii) $\psi(x) = 1$ for $|x| \le 1$.

Hint: First construct a positive smooth function $\tilde{\chi} : \mathbb{R} \to [0, 1]$ such that

$$\tilde{\chi}(t) = \begin{cases} 0 & t < -1 \\ 1 & t > 1 \end{cases}$$

Solution We define $\psi_0 : \mathbb{R} \to \mathbb{R}$ by:

$$\psi_0(x) = \chi\left(1 - x^2\right)$$

with χ as in the previous part. This is smooth and positive, and supported in $[-1, 1]$. Now set:

$$\tilde{\chi}(t) = \frac{\int_{-\infty}^{t} \chi(s)ds}{\int_{-\infty}^{\infty} \chi(s)ds}.$$

This satisfies the conditions of the hint. Now finally set $\psi(x) = \tilde{\chi}(3 - 2|x|^2)$.

Exercise 1.5. a) Suppose $\phi \in \mathscr{E}(\mathbb{R}^n)$. Let $\{x_l\}_{l=1}^{\infty} \subset \mathbb{R}^n$ be a sequence with $x_l \to 0$. Show that

$$\tau_{x_l}\phi \to \phi, \qquad \text{as } l \to \infty.$$

in $\mathscr{E}(\mathbb{R}^n)$, where τ_x is the translation operator defined in equation (1.2).

Solution Fix a compact $K \subset B_R(0)$. $D^\alpha\phi$ is continuous on $\overline{B_{R+1}(0)}$, hence uniformly continuous. Fix $\epsilon > 0$. By the uniform continuity of $D^\alpha\phi$ there exists $\delta > 0$ such that for any $x, y \in \overline{B_{R+1}(0)}$ with $|x - y| < \delta$, we have:

$$|D^\alpha\phi(y) - D^\alpha\phi(x)| < \epsilon.$$

There exists L such that for all $l \geq L$ we have $|x_l| < \min\{\delta, 1\}$, so that for any $x \in K$ we have: $x - x_l \in \overline{B_{R+1}(0)}$ and $|x_l| < \delta$. Thus:

$$|D^\alpha\phi(x - x_l) - D^\alpha\phi(x)| < \epsilon.$$

Since this holds for any $x \in K$, we have:

$$\sup_{x \in K} |D^\alpha\phi(x - x_l) - D^\alpha\phi(x)| < \epsilon.$$

This is precisely the statement that $D^\alpha\tau_{x_l}\phi \to D^\alpha\phi$ uniformly in K. Since we have uniform convergence of $D^\alpha\tau_{x_l}\phi$ on arbitrary compact subsets, we have convergence in $\mathscr{E}(\mathbb{R}^n)$.

b) Suppose $\phi \in \mathscr{E}(\mathbb{R}^n)$, show that

$$\Delta_i^h\phi \to D_i\phi, \qquad \text{as } h \to 0,$$

in $\mathscr{E}(\mathbb{R}^n)$, where Δ_i^h is the difference quotient defined in Example 2.

Solution Fix a compact $K \subset B_R(0)$, and suppose that $|h| < 1$. We have by the mean value theorem that for each $x \in K$ there exists s_x with $|s_x| < |h|$ such that:

$$D^\alpha\Delta_i^h\phi(x) = \frac{D^\alpha\phi(x + he_i) - D^\alpha\phi(x)}{h} = D_i D^\alpha\phi(x + s_x e_i)$$

$D_iD^\alpha\phi$ is continuous on $\overline{B_{R+1}(0)}$, hence uniformly continuous. Fix $\epsilon > 0$. By the uniform continuity of $D^\alpha\phi$ there exists $\delta > 0$ such that for any $x, y \in \overline{B_{R+1}(0)}$ with $|x-y| < \delta$, we have:

$$|D_iD^\alpha\phi(y) - D_iD^\alpha\phi(x)| < \epsilon.$$

Take $|h| < \min\{\delta, 1\}$. Then for any $x \in K$ we have $x + s_x e_i \in \overline{B_{R+1}(0)}$ and $|s_x e_i| = |s_x| < \delta$. Thus

$$\left| D^\alpha \Delta_i^h \phi(x) - D^\alpha D_i\phi(x) \right| = |D_iD^\alpha\phi(x + s_x e_i) - D_iD^\alpha\phi(x)| < \epsilon$$

We thus conclude that for h sufficiently small,

$$\sup_{x\in K} \left| D^\alpha \Delta_i^h \phi(x) - D^\alpha D_i\phi(x) \right| < \epsilon.$$

Which implies that we have uniform convergence of $D^\alpha\Delta_i^h\phi$ on arbitrary compact sets, thus convergence in $\mathscr{E}(\mathbb{R}^n)$.

Exercise 1.6. a) Show that $\mathscr{S}$ is a vector subspace of $\mathscr{E}(\mathbb{R}^n)$. Show that if $\{\phi_j\}_{j=1}^\infty$ is a sequence of rapidly decreasing functions which tends to zero in $\mathscr{S}$, then $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$.

Solution To see that $\mathscr{S}$ is a vector subspace it's enough to verify that $\phi_1 + \lambda\phi_2 \in \mathscr{S}$ whenever $\phi_1, \phi_2 \in \mathscr{S}$ and $\lambda \in \mathbb{C}$. For any multi-index α and $N \in \mathbb{N}$ we have:

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \left[\phi_1 + \lambda\phi_2\right](x)\right| &\le \sup_{x\in\mathbb{R}^n} \left[\left|(1+|x|)^N D^\alpha\phi_1\right| + |\lambda| \left|(1+|x|)^N D^\alpha\phi_2\right|\right] \\ &\le \sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha\phi_1(x)\right| \\ &+ |\lambda| \sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha\phi(x)\right| < \infty\end{aligned}$$

If $\phi_j \to 0$ in $\mathscr{S}$, then in particular we have that $D^\alpha\phi_j \to 0$ uniformly in $\mathbb{R}^n$ (setting $N = 0$). Thus $D^\alpha\phi_j \to 0$ uniformly in any compact subset of $\mathbb{R}^n$, so that $D^\alpha\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$.

b) Show that $\mathscr{D}(\mathbb{R}^n)$ is a vector subspace of $\mathscr{S}$. Show that if $\{\phi_j\}_{j=1}^\infty$ is a sequence of compactly supported functions which tends to zero in $\mathscr{D}(\mathbb{R}^n)$ then $\phi_j \to 0$ in $\mathscr{S}$.

Solution Clearly if ϕ has compact support, then $\phi \in \mathscr{S}$. Since $\operatorname{supp}\phi_1 + \lambda\phi_2 \subset \operatorname{supp}\phi_1 + \operatorname{supp}\phi_2$, we have that $\mathscr{D}(\mathbb{R}^n)$ is closed and hence a subspace of $\mathscr{S}$. If $\phi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$, then there exists R such that $\operatorname{supp}\phi_j \subset B_R(0)$. We then have that:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha\phi_j(x)\right| \le (1+R)^N \sup_{x\in B_R(0)} |D^\alpha\phi_j(x)| \to 0,$$

since we know that $D^\alpha\phi_j \to 0$ uniformly.

c) Give an example of a sequence $\{\phi_j\}_{j=1}^{\infty} \subset C_0^{\infty}(\mathbb{R}^n)$ such that

i) $\phi_j \to 0$ in $\mathscr{S}$, but ϕ_j has no limit in $\mathscr{D}(\mathbb{R}^n)$.

Solution We can take, for example:

$$\phi_j(x) = e^{-j^2}\phi\left(x - je_i\right)$$

for some (non-zero) $\phi \in \mathscr{D}(\mathbb{R}^n)$.

ii) $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$, but ϕ_j has no limit in $\mathscr{S}$.

Solution We can take, for example:

$$\phi_j(x) = \phi\left(x - je_i\right)$$

for some (non-zero) $\phi \in \mathscr{D}(\mathbb{R}^n)$.

Exercise 1.7. a) Suppose $\phi \in \mathscr{S}$. Let $\{x_l\}_{l=1}^{\infty} \subset \mathbb{R}^n$ be a sequence with $x_l \to 0$. Show that

$$\tau_{x_l}\phi \to \phi, \qquad \text{as } l \to \infty.$$

in $\mathscr{S}$, where τ_x is the translation operator defined in equation (1.2).

Solution Fix $N \in \mathbb{N}$ and α a multi-index. We note that, by the fundamental theorem of Calculus, we have:

$$\begin{aligned} |D^{\alpha}\tau_{x_l}\phi(x) - D^{\alpha}\phi(x)| &= \left| \int_0^1 \frac{d}{dt} D^{\alpha}\phi(x - x_l t)dt \right| \\ &= \left| \int_0^1 x_l \cdot DD^{\alpha}\phi(x - x_l t)dt \right| \\ &\leq |x_l| \sup_{t\in(0,1)} |DD^{\alpha}\phi(x - x_l t)| \, . \end{aligned}$$

Now, recall that since $\phi \in \mathscr{S}$, there exists a constant C (depending on N, α) such that:

$$|D_i D^{\alpha}\phi(x - x_l t)| \leq \frac{C}{(1 + |x - x_l t|)^N}$$

Now, if $|x_l| < 1$, then:

$$1 + |x - x_l t| \geq 1 + |x| - |x_l|\, t \geq \frac{1}{2} + |x| \geq \frac{1}{2}(1 + |x|).$$

Thus for $|x_l| < 1$, we have

$$|D_i D^{\alpha}\phi(x - x_l t)| \leq \frac{2^N C}{(1 + |x|)^N}.$$

We conclude that:

$$(1+|x|)^N |D^\alpha \tau_{x_l}\phi(x) - D^\alpha \phi(x)| \le |x_l| N 2^N C$$

Since $x_l \to 0$ we conclude that:

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^N |D^\alpha \tau_{x_l}\phi(x) - D^\alpha \phi(x)| \to 0$$

and thus $\tau_{x_l}\phi \to \phi$ in $\mathscr{S}$.

b) Suppose $\phi \in \mathscr{S}$, show that

$$\Delta_i^h \phi \to D_i\phi, \qquad \text{as } h \to 0,$$

in $\mathscr{S}$, where Δ_i^h is the difference quotient defined in Example 2.

Solution Fix $N \in \mathbb{N}$ and α a multi-index. We have by the mean value theorem that for each $x \in \mathbb{R}^n$ there exists s_x with $|s_x| < |h|$ such that:

$$D^\alpha \Delta_i^h \phi(x) = \frac{D^\alpha \phi(x+he_i) - D^\alpha \phi(x)}{h} = D_i D^\alpha \phi(x + s_x e_i)$$

Therefore,

$$\left| D^\alpha \Delta_i^h \phi(x) - D^\alpha D_i \phi(x) \right| = |D_i D^\alpha \phi(x+s_x e_i) - D_i D^\alpha \phi(x)|$$

Again, as in the previous part, we can make use of the fundamental theorem of calculus to deduce that:

$$\begin{aligned} \left| D^\alpha \Delta_i^h \phi(x) - D^\alpha D_i \phi(x) \right| &\le |s_x| \sup_{t\in(0,1)} |D_i D_i D^\alpha \phi(x + t s_x e_i)| \\ &\le |h| \sup_{t\in(0,1)} |D_i D_i D^\alpha \phi(x + t h e_i)| \end{aligned}$$

An argument precisely as for the previous part shows that as $h \to 0$:

$$\sup_{x\in\mathbb{R}^n} (1+|x|)^N \left| D^\alpha \Delta_i^h \phi(x) - D^\alpha D_i \phi(x) \right| \to 0.$$

so that $\Delta_i^h \phi(x) \to D_i\phi(x)$ in $\mathscr{S}$.

Exercise 2.1 (*). Suppose that we work over $\mathbb{R}^n$ and that $f, g, h \in \mathscr{S}$.

a) Show that for any multi-index α, we have that $D^\alpha f \in L^p(\mathbb{R}^n)$ for $1 \le p < \infty$, i.e. that

$$||D^\alpha f||_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Solution Note that $f \in \mathscr{S}$ in particular implies that for any N and multi-index α there exists a constant $C_{\alpha,N}$ such that for $x \in \mathbb{R}^n$ we have:

$$|f(x)| \le \frac{C_{\alpha,N}}{(1+|x|)^N}.$$

Now, we estimate:

$$\begin{aligned} ||D^\alpha f||^p_{L^p(\mathbb{R}^n} &= \int_{\mathbb{R}^n} |D^\alpha f(x)|^p \, dx \\ &\le (C_{\alpha,N})^p \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{Np}} dx \\ &\le (C_{\alpha,N})^p \, \sigma_{n-1} \int_0^\infty \frac{1}{(1+r)^{Np}} r^{n-1} dr \end{aligned}$$

Setting $N > p^{-1}(n+1)$, we have

$$\int_0^\infty \frac{1}{(1+r)^{Np}} r^{n-1} dr \le \int_0^\infty \frac{1}{(1+r)^{n+1}} r^{n-1} dr = \frac{1}{n}\left[\frac{r^n}{(1+r)^n}\right]_0^\infty = \frac{1}{n},$$

so that

$$||D^\alpha f||_{L^p(\mathbb{R}^n} < \infty.$$

b) Define

$$\begin{aligned} F \ &: \ \mathbb{R}^n \times \mathbb{R}^n, \\ & \quad (x,y) \mapsto f(x)g(y-x). \end{aligned}$$

Show that $F \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

Solution First note that

$$|x| + |y-x| \ge |x| + \frac{1}{2}|y-x| \ge \frac{1}{2}(|x|+|y|)$$

furthermore:

$$\frac{1}{1+|x|}\frac{1}{1+|z|} = \frac{1}{1+|x|+|y|+|x|\,|y|} \le \frac{1}{1+|x|+|y|}$$

Combining these two facts, we deduce that

$$\frac{1}{(1+|x|)^N}\frac{1}{(1+|y-x|)^N} \le \frac{2^N}{1+|x|+|y|}.$$

Now, let $Z=(x,y)\in\mathbb{R}^{2n}$. We have, using the fact that $f,g\in\mathscr{S}$:

$$\begin{aligned}
|F(Z)| &= |f(x)g(y-x)| \\
&\le \frac{C_N}{(1+|x|)^N}\frac{C_N'}{(1+|y-x|)^N} \\
&\le C_N C_N' \frac{2^N}{1+|x|+|y|} \\
&= 2^N C_N C_N' \frac{1}{(1+|Z|)^N}.
\end{aligned}$$

By taking N large enough, we can show using the same argument as in the previous part that $F\in L^1(\mathbb{R}^{2n})$.

c) For each $x\in\mathbb{R}^n$, set

$$\begin{aligned}
G_x \quad : \quad & \mathbb{R}^n\times\mathbb{R}^n, \\
& (y,z)\mapsto f(y)g(z)h(x-y-z).
\end{aligned}$$

Show that $G_x\in L^1(\mathbb{R}^n\times\mathbb{R}^n)$.

Solution As in the previous part, we have:

$$|g(z)h(x-y-z)| \le \frac{C_N}{(1+|z|+|x-y|)^N} \le \frac{C_N'}{(1+|z|+|x|+|y|)^N}$$

So that, setting $Z=(y,z)$, and using $|f|<C$, we have:

$$|G_z(Z)| \le \frac{C_N''}{(1+|z|+|x|+|y|)^N} \le \frac{C_N'}{(1+|Z|)^N}$$

for any N and by a similar argument to previously, $G_z\in L^1(\mathbb{R}^{2n})$.

Exercise 2.2. Show that Theorem 1.10 holds under the alternative hypotheses:

a) $f\in L^1(\mathbb{R}^n)$, $g\in C^k(\mathbb{R}^n)$ with $\sup_{\mathbb{R}^n}|D^\alpha g|<\infty$ for all $|\alpha|\le k$.

Solution 1. First we establish the result for $k=0$. We need to show that if $f\in L^1(\mathbb{R}^n)$ and $g\in C^0(\mathbb{R}^n)$ is bounded then $f\star g$ is continuous. To show this, it suffices to show that $f\star g(x-z_j)\to f\star g(x)$ for any sequence $\{z_j\}_{j=1}^\infty$ with $z_j\to 0$. Now, note that

$$f\star g(x-z_j) = \int_{\mathbb{R}^n} f(y)g(x-z_j-y)dy = \int_{\mathbb{R}^n} f(y)\tau_{z_j}g(x-y)dy.$$

Now, sending $j \to \infty$, we are done, so long as we can justify interchanging the limit and the integral. Note that for any fixed x and all j:

$$\left|f(y)\tau_{z_j} g(x-y)\right| \leq \sup_{\mathbb{R}^n} |g| \, |f(y)|$$

Since $f \in L^1(\mathbb{R}^n)$ and g is bounded the right hand side is integrable, and so by the dominated convergence theorem:

$$\lim_{j\to\infty} f \star g(x - z_j) = \int_{\mathbb{R}^n} \lim_{j\to\infty} f(y)\tau_{z_j} g(x-y)dy = \int_{\mathbb{R}^n} f(y)g(x-y)dy = f \star g(x).$$

2. Now suppose that $f \in L^1(\mathbb{R}^n)$ and $g \in C^1(\mathbb{R}^n)$ with g, $D_j g$ bounded. Clearly $f \star D_i g$ is continuous by the previous argument. To show $f \star g \in C^1(\mathbb{R}^n)$, it suffices to show that for any $x \in \mathbb{R}^n$ and any sequence $\{h_j\}_{j=1}^\infty \subset \mathbb{R}$ with $h_j \to 0$ we have:

$$\lim_{j\to\infty} \Delta_i^{h_j} f \star g(x) = f \star D_i g(x).$$

Note that

$$\begin{aligned}
\Delta_i^{h_j} f \star g(x) &= \frac{f \star g(x + h_j e_i) - f \star g(x)}{h} \\
&= \int_{\mathbb{R}^n} f(y) \left(\frac{g(x + h_j e_i - y) - g(x-y)}{h} \right) dy \\
&= \int_{\mathbb{R}^n} f(y) \Delta_i^{h_j} g(x-y) dy
\end{aligned}$$

so that again we are done provided we can send $j \to \infty$ and interchange the limit and the integral. We note that:

$$\Delta_i^{h_j} g(x) = \frac{1}{h_j}\int_0^{h_j} \frac{d}{dt}\left[g(x + te_i)\right] dt = \frac{1}{h_j}\int_0^{h_j} D_i g(x + te_i) dt$$

so we can estimate:

$$\left|\Delta_i^{h_j} g(x)\right| \leq \sup_{\mathbb{R}^n} |D_i g(x)|$$

An argument precisely analogous to the previous part permits us to apply DCT and conclude that:

$$\lim_{j\to\infty} \Delta_i^{h_j} f \star g(x) = \int_{\mathbb{R}^n} \lim_{j\to\infty} f(y)\Delta_i^{h_j} g(x-y)dy = f \star D_i g(x).$$

3. The case where $g \in C^k(\mathbb{R}^n)$ with $k > 1$ now follows by a simple induction.

b) $f \in L^1(\mathbb{R}^n)$ with $\operatorname{supp} f$ compact, $g \in C^k(\mathbb{R}^n)$.

Solution 1. First we establish the result for $k = 0$. We need to show that if $f \in L^1(\mathbb{R}^n)$ with compact support and $g \in C^k(\mathbb{R}^n)$ then $f \star g$ is continuous. To show this, it suffices to show that $f \star g(x - z_j) \to f \star g(x)$ for any sequence $\{z_j\}_{j=1}^{\infty}$ with $z_j \to 0$. Now, note that

$$f \star g(x - z_j) = \int_{\mathbb{R}^n} f(y) g(x - z_j - y) dy = \int_{\mathbb{R}^n} f(y) \tau_{z_j} g(x - y) dy.$$

Now, sending $j \to \infty$, we are done, so long as we can justify interchanging the limit and the integral. Note that for any fixed x and all j:

$$\left| f(y) \tau_{z_j} g(x - y) \right| \leq \left(\sup_{B_R(x)} |g| \right) |f(y)|$$

where R is sufficiently large that $\operatorname{supp} f + h_j \subset B_R(0)$ for all j. Since $f \in L^1(\mathbb{R}^n)$ the right hand side is integrable, and so by the dominated convergence theorem:

$$\lim_{j \to \infty} f \star g(x - z_j) = \int_{\mathbb{R}^n} \lim_{j \to \infty} f(y) \tau_{z_j} g(x - y) dy = \int_{\mathbb{R}^n} f(y) g(x - y) dy = f \star g(x).$$

2. Now suppose that $f \in L^1(\mathbb{R}^n)$ has compact support and $g \in C^1(\mathbb{R}^n)$. Clearly $f \star D_i g$ is continuous by the previous argument. To show $f \star g \in C^1(\mathbb{R}^n)$, it suffices to show that for any $x \in \mathbb{R}^n$ and any sequence $\{h_j\}_{j=1}^{\infty} \subset \mathbb{R}$ with $h_j \to 0$ we have:

$$\lim_{j \to \infty} \Delta_i^{h_j} f \star g(x) = f \star D_i g(x).$$

Note that

$$\begin{aligned} \Delta_i^{h_j} f \star g(x) &= \frac{f \star g(x + h_j e_i) - f \star g(x)}{h} \\ &= \int_{\mathbb{R}^n} f(y) \left(\frac{g(x + h_j e_i - y) - g(x - y)}{h} \right) dy \\ &= \int_{\mathbb{R}^n} f(y) \Delta_i^{h_j} g(x - y) dy \end{aligned}$$

so that again we are done provided we can send $j \to \infty$ and interchange the limit and the integral. By the argument in the proof of part a) which allows us to bound the difference quotient by the derivative, we have:

$$\left| f(y) \Delta_i^{h_j} g(x - y) \right| \leq \left(\sup_{B_R(x)} |D_i g| \right) |f(y)|$$

this allows us to invoke the DCT and deduce that:

$$\lim_{j \to \infty} \Delta_i^{h_j} f \star g(x) = \int_{\mathbb{R}^n} \lim_{j \to \infty} f(y) \Delta_i^{h_j} g(x - y) dy = f \star D_i g(x).$$

3. The case where $g \in C^k(\mathbb{R}^n)$ with $k > 1$ now follows by a simple induction.

Exercise 2.3. a) Prove the following identities for $r, s > 0$ and $x \in \mathbb{R}^n$:

i) $B_r(x) + B_s(0) = B_{r+s}(x)$
ii) $\overline{B_r(x)} + B_s(0) = B_{r+s}(x)$
iii) $\overline{B_r(x)} + \overline{B_s(0)} = \overline{B_{r+s}(x)}$

Solution i) Suppose we have $v = z + w$ where $z \in B_r(x)$ and $w \in B_s(0)$, then $|x - z| < r$ and $|w| < s$. Thus:

$$|v - x| = |z + w - x| \leq |z - x| + |w| < r + s.$$

Thus $B_r(x) + B_s(0) \subset B_{r+s}(x)$. Conversely, suppose that $v \in B_{r+s}(x)$. Then $|v - x| = l < r + s$. Define:

$$z = x + \frac{r}{r+s}(v - x), \qquad w = \frac{s}{r+s}(v - x).$$

We have $|z - x| < r$ and $|w| < s$ and $v = z + w$. Thus $v \in B_r(x) + B_s(0)$ and $B_{r+s}(x) \subset B_r(x) + B_s(0)$.

ii) Suppose we have $v = z + w$ where $z \in \overline{B_r(x)}$ and $w \in B_s(0)$, then $|x - z| \leq r$ and $|w| < s$. Thus:

$$|v - x| = |z + w - x| \leq |z - x| + |w| < r + s.$$

Thus $\overline{B_r(x)} + B_s(0) \subset B_{r+s}(x)$. Conversely, suppose that $v \in B_{r+s}(x)$. Then $|v - x| = l < r + s$. Define:

$$z = x + \frac{r}{r+s}(v - x), \qquad w = \frac{s}{r+s}(v - x).$$

We have $|z - x| < r$ and $|w| < s$ and $v = z + w$. Thus $v \in \overline{B_r(x)} + B_s(0)$ and $B_{r+s}(x) \subset \overline{B_r(x)} + B_s(0)$.

iii) Suppose we have $v = z + w$ where $z \in \overline{B_r(x)}$ and $w \in \overline{B_s(0)}$, then $|x - z| \leq r$ and $|w| \leq s$. Thus:

$$|v - x| = |z + w - x| \leq |z - x| + |w| \leq r + s.$$

Thus $\overline{B_r(x)} + \overline{B_s(0)} \subset \overline{B_{r+s}(x)}$. Conversely, suppose that $v \in \overline{B_{r+s}(x)}$. Then $|v - x| = l \leq r + s$. Define:

$$z = x + \frac{r}{r+s}(v - x), \qquad w = \frac{s}{r+s}(v - x).$$

We have $|z - x| \leq r$ and $|w| \leq s$ and $v = z + w$. Thus $v \in \overline{B_r(x)} + \overline{B_s(0)}$ and $\overline{B_{r+s}(x)} \subset \overline{B_r(x)} + \overline{B_s(0)}$

Suppose that $A, B \subset \mathbb{R}^n$. Show that:

b) If one of A or B is open, then so is $A + B$.

Solution Suppose A is open, and that $v = A + B$. We must have $v = a + b$ for some $a \in A$ and $b \in B$. Since A is open, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset A$. Thus for any x with $|x - a| < \epsilon$ we know $x \in A$. Therefore we have:

$$A + B \supset \{x + b : |x - a| < \epsilon\} = \{z : |z - (a + b)| < \epsilon\} = B_\epsilon(a + b)$$

Thus $A + B$ is open, since it contains an open ball about any point.

c) If A and B are both bounded, then so is $A + B$.

Solution

Since A and B are both bounded, there exists $R > 0$ such that $A \subset B_R(0)$ and $B \subset B_R(0)$. If $a \in A$ and $b \in B$ we find:

$$|a + b| \leq |a| + |b| < R + R$$

so that $A + B \subset B_{2R}(0)$.

d) If A is closed and B is compact, then $A + B$ is closed.

Solution Suppose $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ is a Cauchy sequence in $A + B$. By the completeness of $\mathbb{R}^n$, we know that x_j converges, to a point $x \in \mathbb{R}^n$. To establish that $A + B$ is closed, we need to show that $x \in A + B$. For each x_j, there exist a_j, b_j such that:

$$x_j = a_j + b_j.$$

Since B is compact, we can find a convergent subsequence $\{b_{j_k}\}_{k=1}^\infty \subset B$ such that $\lim_{k\to\infty} b_{j_k} = b \in B$. Now consider the sequence:

$$a_{j_k} = x_{j_k} - b_{j_k}$$

This is Cauchy, since both b_{j_k} and x_{j_k} are convergent, hence since A is closed, $a_{j_k} \to a \in A$. We therefore have that:

$$x = \lim_{j\to\infty} x_j = lim_{k\to\infty} x_{j_k} = a + b$$

with $a \in A$ and $b \in B$, thus $x \in A + B$.

e) If A and B are both compact, then so is $A + B$.

Solution Combining the two previous results, we know that the sum of two closed and bounded sets is closed and bounded. We're done by Heine-Borel.

Exercise 2.4. Show that if $f \in C_0^k(\mathbb{R}^n)$ and $g \in C_0^l(\mathbb{R}^n)$ then $f \star g \in C_0^{k+l}(\mathbb{R}^n)$. Conclude that $\mathscr{D}(\mathbb{R}^n)$ is closed under convolution.

Solution If α is a multi-index with $|\alpha| \leq k + l$, we can write $\alpha = \alpha_1 + \alpha_2$ with $|\alpha_1| \leq k$ and $\alpha_2 \leq l$. We then have (applying Theorem 1.10 twice as well as and the symmetry of the convolution):

$$D^\alpha(f \star g) = D^{\alpha_1} D^{\alpha_2}(f \star g) = D^{\alpha_1}(f \star D^{\alpha_2} g) = D^{\alpha_1} f \star D^{\alpha_2} g.$$

Since both functions have compact support, the convolution also has compact support and we're done.

Exercise 2.5 (*)**.** Suppose that $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a positive integrable simple function,

a) Show that Minkowski's integral inequality holds for the case $p = 1$:

$$\int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} F(x,y)dx \right| dy \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |F(x,y)| \, dydx$$

b) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, consider the function $\log\left[ta^p + (1-t)b^q\right]$ and use the concavity of the logarithm

c) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||f||_p = 1$ and $||g||_q = 1$ then

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \leq 1.$$

Deduce Hölder's inequality:

$$\int_{\mathbb{R}^n} |f(x)g(x)| \, dx \ \leq ||f||_p \, ||g||_q \, , \qquad \text{for all } f \in L^p(\mathbb{R}^n), \quad g, \in L^q(\mathbb{R}^n).$$

d) Set $G(y) = \left(\int_{\mathbb{R}^n} F(x,y)dx\right)^{p-1}$

i) Show that if $q = \frac{p}{p-1}$:

$$||G||_{L^q(\mathbb{R}^n)} = \left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p-1}$$

ii) Show that:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p} = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} G(y)F(x,y)dy \right) dx$$

iii) Applying Hölder's inequality, deduce:

$$\left|\left| \int_{\mathbb{R}^n} F(x,\cdot)dx \right|\right|_{L^p(\mathbb{R}^n)}^{p} \leq ||G||_{L^q(\mathbb{R}^n)} \int_{\mathbb{R}^n} ||F(x,\cdot)||_{L^p(\mathbb{R}^n)} \, dx$$

e) Deduce that Minkowski's integral inequality

$$\left[\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}F(x,y)dx\right|^p dy\right]^{\frac{1}{p}} \leq \int_{\mathbb{R}^n}\left[\int_{\mathbb{R}^n}|F(x,y)|^p\,dy\right]^{\frac{1}{p}}dx$$

holds for any measurable function $F:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$, where $1\leq p<\infty$.

Exercise 2.6. a) Show that δ_x, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

Solution Let $x\in\Omega$, where $\Omega\subset\mathbb{R}^n$ is open. Suppose $\phi_1,\phi_2\in\mathscr{D}(\Omega)$ and $a\in\mathbb{C}$. Then:

$$\delta_x[\phi_1+a\phi_2]=\phi_1(x)+a\phi_2(x)=\delta_x[\phi_1]+a\delta_x[\phi_2]$$

Thus $\delta_x:\mathscr{D}(\Omega)\to\mathbb{C}$ is linear. Fix a compact $K\subset\Omega$. We certainly have:

$$|\delta_x[\phi]|=|\phi(x)|\leq\sup_{x\in K}|\phi(x)|$$

for all $\phi\in C_0^\infty(K)$. Thus δ_x is continuous. Since we can control $\delta_x[\phi]$ by $|\phi(x)|$ without any derivatives on any K, we have that δ_x **is order** 0.

b) Show that T_f, as defined in Example 7 is continuous and linear, hence a distribution. Find the order.

Solution Let $\Omega\subset\mathbb{R}^n$ is open. Suppose $\phi_1,\phi_2\in\mathscr{D}(\Omega)$ and $a\in\mathbb{C}$. Suppose $f\in L^1_{loc.}(\Omega)$. Then:

$$\begin{aligned}T_f[\phi_1+a\phi_2]&=\int_\Omega f(x)\left(\phi_1(x)+a\phi_2(x)\right)dx\\&=\int_\Omega f(x)\phi_1(x)dx+a\int_\Omega f(x)\phi_2(x)dx=T_f[\phi_1]+aT_f[\phi_2].\end{aligned}$$

Now, suppose $K\subset\Omega$ is compact and $\phi\in C_0^\infty(K)$. Then:

$$|T_f[\phi]|=\left|\int_\Omega f(x)\phi(x)dx\right|\leq\sup_{y\in K}|\phi(y)|\left|\int_\Omega f(x)dx\right|=C_K\sup_{y\in K}|\phi(y)|$$

we conclude that T_f is continuous, and has order 0.

c) By constructing a suitable sequence of smooth functions show that the order of $P.V.\left(\frac{1}{x}\right)$ is one.

Solution Recall that for $\phi\in\mathscr{D}(\mathbb{R})$ we have:

$$P.V.\left(\frac{1}{x}\right)[\phi]=\lim_{\epsilon\to 0}\left[\int_{-\infty}^{-\epsilon}\frac{\phi(x)}{x}dx+\int_\epsilon^\infty\frac{\phi(x)}{x}dx\right].$$

We saw in lectures that if $\phi \in C_0^\infty(B_R(0))$ then:

$$\left| P.V. \left(\frac{1}{x}\right) [\phi] \right| \leq 4R \sup_{\mathbb{R}^n} |\phi'| .$$

To show that the order of $P.V. \left(\frac{1}{x}\right)$ is one we need to establish that there exists no constant C_R such that:

$$\left| P.V. \left(\frac{1}{x}\right) [\phi] \right| \leq C_R \sup_{\mathbb{R}^n} |\phi| . \tag{1}$$

holds for all $\phi \in C_0^\infty(B_R(0))$. Suppose $\delta \in (0, \frac{1}{2})$ and let $\psi_\delta \in C_c^\infty(\mathbb{R})$ be a function satisfying:

i) $0 \leq \psi_\delta \leq 1$.
ii) $\psi_\delta(x) = 0$ for $x < \delta$ and $x > 2$.
iii) $\psi_\delta(x) = 1$ for $2\delta < x < 1$

We've seen in lectures how such a function may be constructed. We have that:

$$\left| P.V. \left(\frac{1}{x}\right) [\phi] \right| = \int_\delta^2 \frac{\psi_\delta(x)}{x} dx \geq \int_{2\delta}^1 \frac{1}{x} dx = \frac{1}{4\delta^2} - 1.$$

Suppose that (**??**) holds for some C_R. Then we conclude that:

$$\frac{1}{4\delta^2} - 1 \leq C_2$$

holds for all $\delta \in (0, \frac{1}{2})$. Clearly this is absurd, so $P.V. \left(\frac{1}{x}\right)$ has order 1.

Exercise 2.7. Suppose $f \in L^1_{loc.}(\Omega)$. Take ϕ_ϵ as in Theorem 1.13, and define for x with $d(x, \partial\Omega) \geq \epsilon$:

$$f_\epsilon(x) := T_f \left[\tau_x \check{\phi}_\epsilon\right] .$$

Show that for any compact $K \subset \Omega$:

$$||f_\epsilon - f||_{L^1(K)} \to 0$$

as $\epsilon \to 0$.
[Hint: follow the proof of Theorem 2.2, but use part b) of Theorem 1.13]

Solution Fix $K \subset \Omega$ compact. Recall that by Lemma 1.14, there exists $\chi \in C_0^\infty(\Omega)$ such that $\chi = 1$ on $K + B_\delta(0)$ for some $\delta > 0$. Take $\epsilon < \delta$. Then, since $\operatorname{supp} \phi_\epsilon \subset B_\epsilon(0)$ we have for $x \in K$:

$$\begin{aligned} f_\epsilon(x) &= \int_\Omega f(y) \tau_x \check{\phi}_\epsilon(y) dy \\ &= \int_\Omega f(y) \phi_\epsilon(x - y) dy \\ &= \int_{\mathbb{R}^n} \chi(y) f(y) \phi_\epsilon(x - y) dy \\ &= \phi_\epsilon \star (\chi f). \end{aligned}$$

Here we have used the fact that when $|x-y|<\epsilon$ we have $\chi = 1$ to insert the cut-off function without altering the integral. Now, since χ is smooth, and $f \in L^1_{loc.}\Omega$, we have $\chi f \in L^1(\mathbb{R}^n)$, so as $\epsilon \to 0$, we have by Theorem 1.13 that:

$$||f_\epsilon - \chi f||_{L^1(\mathbb{R}^n)} \to 0$$

Now since $\chi = 1$ on K we deduce:

$$||f_\epsilon - f||_{L^1(K)} = ||f_\epsilon - \chi f||_{L^1(K)} \leq ||f_\epsilon - \chi f||_{L^1(\mathbb{R}^n)} \to 0$$

which is the result we require.

Exercise 2.8. a) Show that if $f_1, f_2 \in C^0(\Omega)$ and $a \in C^\infty(\Omega)$, then

$$aT_{f_1} + T_{f_2} = T_{af_1+f_2}$$

Solution Let $\phi \in \mathscr{D}(\Omega)$. We calculate:

$$\begin{aligned}
(aT_{f_1} + T_{f_2})[\phi] &= aT_{f_1}[\phi] + T_{f_2}[\phi] \\
&= T_{f_1}[a\phi] + T_{f_2}[\phi] \\
&= \int_\Omega f_1(x)a(x)\phi(x)dx + \int_\Omega f_2(x)\phi(x)dx \\
&= \int_\Omega (f_1(x)a(x) + f_2(x))\,\phi(x)dx = T_{af_1+f_2}[\phi].
\end{aligned}$$

Since ϕ is arbitrary the result follows.

b) Show that if $f \in C^k(\Omega)$ then

$$D^\alpha T_f = T_{D^\alpha f}$$

for $|\alpha| \leq k$. Deduce that $\iota \circ D^\alpha = D^\alpha \circ \iota$.

Solution Let $\phi \in \mathscr{D}(\Omega)$. We calculate:

$$\begin{aligned}
D^\alpha T_f[\phi] &= (-1)^{|\alpha|}\, T_f\,[D^\alpha \phi] \\
&= (-1)^{|\alpha|} \int_\Omega f(x) D^\alpha \phi(x)dx \\
&= \int_\Omega D^\alpha f(x)\phi(x)dx = T_{D^\alpha f}[\phi]
\end{aligned}$$

where we have used the compact support of ϕ to integrate by parts. Since ϕ is arbitrary the result follows.

c) Deduce that if $f \in C^k(\Omega)$ then

$$\sum_{|\alpha|\leq k} a_\alpha D^\alpha T_f = T_{Lf}.$$

where

$$Lf = \sum_{|\alpha|\leq k} a_\alpha D^\alpha f$$

Solution This follows immediately from the two previous results.

Exercise 2.9. a) Show that for $f, g \in C^0_0(\mathbb{R}^n)$:

$$T_{f\star g} = T_f \star T_g.$$

Solution Let $\phi \in \mathscr{D}(\mathbb{R}^n)$. Recall that $T_f \star \phi = f \star \phi$, so:

$$\begin{aligned} T_{f\star g} \star \phi &= (f \star g) \star \phi = f \star (g \star \phi) = T_f \star (g \star \phi) \\ &= T_f \star (T_g \star \phi) = (T_f \star T_g) \star \phi \end{aligned}$$

Here we have used that the convolution of functions is associative, as well as the definition of convolution of distributions.

b) Show that convolution is linear in both of its arguments, i.e. if $u_i \in \mathscr{D}'(\mathbb{R}^n)$ and u_3, u_4 have compact support then

$$(u_1 + au_2) \star u_3 = u_1 \star u_3 + au_2 \star u_3$$

and

$$u_1 \star (u_3 + au_4) = u_1 \star u_3 + au_1 \star u_4$$

where $a \in \mathbb{C}$ is a constant.

Solution First note that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ is a test function, then:

$$\begin{aligned} (u_1 + au_2) \star \phi(x) &= (u_1 + au_2)[\tau_x \check{\phi}] \\ &= u_1[\tau_x \check{\phi}] + au_2[\tau_x \check{\phi}] \\ &= u_1 \star \phi(x) + au_2 \star \phi(x). \end{aligned}$$

Moreover if ϕ_1, ϕ_2 are both test functions, then:

$$\begin{aligned} u \star (\phi_1 + a\phi_2) &= u\left[\tau_x \check{\phi}_1 + a\tau_x \check{\phi}_2\right] \\ &= u\left[\tau_x \check{\phi}_1\right] + au\left[\tau_x \check{\phi}_2\right] \\ &= u \star \phi_1 + au \star \phi_2. \end{aligned}$$

We deduce that:

$$
\begin{aligned}
((u_1 + au_2) \star u_3) \star \phi &= (u_1 + au_2) \star (u_3 \star \phi) \\
&= u_1 \star (u_3 \star \phi) + au_2 \star (u_3 \star \phi) \\
&= (u_1 \star u_3) \star \phi + a(u_2 \star u_3) \star \phi \\
&= (u_1 \star u_3 + au_2 \star u_3) \star \phi.
\end{aligned}
$$

Since ϕ was arbitrary, the result follows. For the second part, we calculate:

$$
\begin{aligned}
(u_1 \star (u_3 + au_4)) \star \phi &= u_1 \star ((u_3 + au_4) \star \phi) \\
&= u_1 \star (u_3 \star \phi + au_4 \star \phi) \\
&= u_1 \star (u_3 \star \phi) + au_1 \star (u_4 \star \phi) \\
&= (u_1 \star u_3) \star \phi + a(u_1 \star u_4) \star \phi \\
&= (u_1 \star u_3 + au_1 \star u_4) \star \phi
\end{aligned}
$$

Since ϕ was arbitrary, the result follows.

Exercise 2.10. a) Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ then

$$\delta_0 \star \phi = \phi$$

Solution

$$\delta_0 \star \phi(x) = \delta_0 \left[\tau_x \check{\phi}\right] = \tau_x \check{\phi}(0) = \phi(x - 0) = \phi(x).$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ has compact support, then

$$\delta_0 \star u = u$$

Solution Let $\phi \in \mathscr{D}(\mathbb{R}^n)$ be arbitrary. Then:

$$(\delta_0 \star u) \star \phi = \delta_0 \star (u \star \phi) = u \star \phi$$

Since ϕ was arbitrary the result follows.

Exercise 3.1. a) Suppose that $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{E}(\Omega)$ is a sequence such that $\phi_j \to \phi$ in $\mathscr{E}(\Omega)$, and $\chi \in \mathscr{D}(\Omega)$. Show that

$$\chi\phi_j \to \chi\phi \quad \text{in } \mathscr{D}(\Omega).$$

Solution Since $\phi_j \to \phi$ in $\mathscr{E}(\Omega)$, we know that for any compact $K \subset \Omega$ and any α we have:

$$\sup_{x\in K} |D^\alpha \phi_j(x) - D^\alpha \phi(x)| \to 0.$$

Now consider $\chi\phi_j$. Clearly $\operatorname{supp}(\chi\phi_j) \subset \operatorname{supp}\chi$, since $\chi\phi_j$ must vanish wherever χ vanishes. Since χ has compact support, there exists a compact K' with $\operatorname{supp}(\chi\phi_j) \subset K'$. Further, by the Leibniz rule, we have (for some α):

$$D^\alpha \chi\phi_j = \sum_{\{\beta:\beta\leq\alpha\}} C_{\alpha,\beta} D^\beta \chi D^{\alpha-\beta}\phi_j,$$

where $C_{\alpha,\beta}$ are some combinatorial constants[1]. Thus:

$$D^\alpha(\chi\phi_j)(x) - D^\alpha(\chi\phi)(x) = \sum_{\{\beta:\beta\leq\alpha\}} C_{\alpha,\beta} D^\beta\chi(x) \left[D^{\alpha-\beta}\phi_j(x) - D^{\alpha-\beta}\phi_j(x)\right].$$

We can then estimate:

$$\begin{aligned}|D^\alpha(\chi\phi_j)(x) - D^\alpha(\chi\phi)(x)| &\leq \sup_{x\in K'} \sum_{\{\beta:\beta\leq\alpha\}} C_{\alpha,\beta} \left|D^\beta\chi(x)\right| \left|D^{\alpha-\beta}\phi_j(x) - D^{\alpha-\beta}\phi_j(x)\right|. \\ &\leq \sum_{\{\beta:\beta\leq\alpha\}} M_{\alpha\beta} \sup_{x\in K'} \left|D^{\alpha-\beta}\phi_j(x) - D^{\alpha-\beta}\phi_j(x)\right|.\end{aligned}$$

Where

$$M_{\alpha\beta} = C_{\alpha,\beta} \sup_{x\in K'} \left|D^\beta\chi(x)\right|.$$

Thus, we have:

$$\sup_{x\in K'} |D^\alpha(\chi\phi_j)(x) - D^\alpha(\chi\phi)(x)| \leq \sum_{\{\beta:\beta\leq\alpha\}} M_{\alpha\beta} \sup_{x\in K'} \left|D^{\alpha-\beta}\phi_j(x) - D^{\alpha-\beta}\phi_j(x)\right|.$$

The right hand side is a finite sum of terms tending to zero. Thus we've established that the support of $\chi\phi_j$ is contained in K', a compact set, for all j, and moreover, we have:

$$\sup_{x\in K'} |D^\alpha(\chi\phi_j)(x) - D^\alpha(\chi\phi)(x)| \to 0.$$

[1] $\beta \leq \alpha$ means $\beta_1 \leq \alpha_1$ and $\beta_2 \leq \alpha_2$, etc.

b) Show that if $\psi \in \mathscr{E}(\Omega)$, then there exists a sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{D}(\Omega)$ such that $\phi_j \to \psi$ in $\mathscr{E}(\Omega)$.
[Hint: Take an exhaustion of Ω by compact sets and apply Lemma 1.14]

Solution Suppose that $\{K_i\}_{i=1}^\infty$ is an exhaustion of Ω by compact sets. That is to say, $K_i \subset \Omega$ is compact and we have:

$$K_i \subset K_{i+1}^\circ, \qquad \bigcup_{i=1}^\infty K_i = \Omega.$$

By Lemma 1.14, for each i we can find a $\chi_i \in \mathscr{D}(\Omega)$ such that $\chi_i(x) = 1$ for all $x \in K_i$. Clearly $\chi_i \psi \in \mathscr{D}(\Omega)$, since $\operatorname{supp}(\chi_i \psi) \subset \operatorname{supp} \chi_i$, which is compact. Now let $E \subset \Omega$ be any fixed compact set. For $i \geq I$, where I is sufficiently large, we have $E \subset K_i$. To see this, note that $\{K_i^\circ\}_{i=1}^\infty$ forms an open cover of Ω and hence of E, so must admit a finite subcover of E. Now, for $i \geq I$, we have that $\chi_i \equiv 1$ on E. Thus:

$$\sup_{x \in E} |D^\alpha(\chi_i \psi) - D^\alpha \psi| = 0$$

so in particular, $D^\alpha(\chi_i \psi) - D^\alpha \psi \to 0$. This is precisely the criterion that $\chi_i \psi \to \psi$ in $\mathscr{E}(\Omega)$.

Exercise 3.2. a) Prove Lemma 2.11.
[Hint: You should argue in a similar fashion to the proof of Lemma 2.9]

Solution First we show that (2.10) implies that u is continuous. Pick a sequence $\{\phi_j\}_{j=1}^\infty \subset \mathscr{S}(\mathbb{R}^n)$ which converges to zero in $\mathscr{S}(\mathbb{R}^n)$. This means that for all α and any $M \geq 0$ we have

$$\sup_{x \in \mathbb{R}^n} \left|(1 + |x|)^M D^\alpha \phi(x)\right| \to 0.$$

In particular, this holds with $M = N$ and for all α with $|\alpha| \leq k$, so as $j \to \infty$ we have:

$$|u[\phi_j]| \leq C \sup_{x \in \mathbb{R}^n; |\alpha| \leq k} \left|(1 + |x|)^N D^\alpha \phi(x)\right| \to 0,$$

and so u is continuous.

To show the opposite implication, we assume that (2.10) does not hold for any k, N, C. Since (2.10) does not hold for any k, N, C, in particular it does not hold for $k = j$, $N = j$ and $C = j$. Thus there must exist $\phi_j \in \mathscr{S}(\mathbb{R}^n)$ such that:

$$|u[\phi_j]| > j \sup_{x \in \mathbb{R}^n; |\alpha| \leq j} \left|(1 + |x|)^j D^\alpha \phi_j(x)\right|.$$

We define:

$$\psi_j(x) = \frac{\phi_j(x)}{|u[\phi_j]|}$$

Clearly $\psi_j \in \mathscr{S}(\mathbb{R}^n)$. We claim that $\psi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$. To see this, fix $M \geq 0$ and a multiindex α. We have, for $j \geq M$ and $j \geq |\alpha|$:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^M D^\alpha \phi_j(x)\right| \leq \sup_{x\in\mathbb{R}^n;|\beta|\leq j} \left|(1+|x|)^j D^\beta \phi_j(x)\right|$$

as a result, we can estimate:

$$\begin{aligned}\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^M D^\alpha \psi_j(x)\right| &= \frac{1}{|u[\phi_j]|} \sup_{x\in\mathbb{R}^n} \left|(1+|x|)^M D^\alpha \phi_j(x)\right| \\ &< \frac{\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^M D^\alpha \phi_j(x)\right|}{j \sup_{x\in K_j;|\beta|\leq j} |D^\beta \phi_j(x)|} < \frac{1}{j}\end{aligned}$$

We conclude that $D^\alpha \psi_j$ tends to zero on $\mathbb{R}^n$, but since α and K were arbitrary, this implies $\psi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$. However, $u[\psi_j] \not\to 0$ since $|u[\psi_j]| = 1$ by construction. Thus u is not continuous. This establishes that if u is continuous, then (2.10) must hold for some k, N, C.

b) Show that we have the inclusions:

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}' \subset \mathscr{D}'(\mathbb{R}^n).$$

Solution Suppose $\{\phi_j\}_{j=1}^\infty \subset \mathscr{D}(\mathbb{R}^n)$ is a sequence such that $\phi_j \to 0$ in $\mathscr{D}(\Omega)$, then there exists an $R > 0$ such that $\operatorname{supp} \phi_j \subset \overline{B_R}(0)$ for all j and moreover:

$$\sup_{x\in\overline{B_R}(0)} |D^\alpha \phi_j(x)| \to 0.$$

for any multi-index α. Now, since functions of compact support are necessarily rapidly decreasing, we have $\{\phi_j\}_{j=1}^\infty \subset \mathscr{S}(\mathbb{R}^n)$. For any $N \geq 0$ and α we can estimate:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \phi_j(x)\right| \leq (1+R)^N \sup_{x\in\overline{B_R}(0)} |D^\alpha \phi_j(x)| \to 0.$$

Now suppose that $u \in \mathscr{S}'$. Since $\mathscr{D}(\mathbb{R}^n) \subset \mathscr{S}$, we can certainly make sense of $u : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ as a linear operator. Moreover, if $\{\phi_j\}_{j=1}^\infty \subset \mathscr{D}(\mathbb{R}^n)$ is a sequence such that $\phi_j \to 0$ in $\mathscr{D}(\Omega)$, then we've just established that $\phi_j \to 0$ in $\mathscr{S}$. As a consequence, we have that $u[\phi_j] \to 0$ by the continuity of $u : \mathscr{S} \to \mathbb{C}$. Thus $u : \mathscr{D}(\mathbb{R}^n) \to \mathbb{C}$ is continuous, so we have shown that $\mathscr{S}' \subset \mathscr{D}'(\mathbb{R}^n)$.

Now suppose $\{\phi_j\}_{j=1}^\infty \subset \mathscr{S}(\mathbb{R}^n)$ is a sequence such that $\phi_j \to 0$ in $\mathscr{S}(\Omega)$, then for any $N \geq 0$ and any multi-index α, we have:

$$\sup_{x\in\mathbb{R}^n} \left|(1+|x|)^N D^\alpha \phi_j(x)\right| \to 0.$$

Since functions in $\mathscr{S}$ are necessarily smooth, we have $\{\phi_j\}_{j=1}^\infty \subset \mathscr{E}(\mathbb{R}^n)$. If K is any compact set, and α any multi-index then:

$$\sup_{x\in K} |D^\alpha \phi_j(x)| \leq \sup_{x\in\mathbb{R}^n} |D^\alpha \phi_j(x)| \to 0.$$

So a $\phi_j \to 0$ in $\mathscr{S}$.

Now suppose $u \in \mathscr{E}'(\mathbb{R}^n)$. Then since $\mathscr{S} \subset \mathscr{E}(\mathbb{R}^n)$, we can certainly make sense of $u : \mathscr{S} \to \mathbb{C}$ as a linear operator. Moreover, if $\{\phi_j\}_{j=1}^\infty \subset \mathscr{S}$ is a sequence such that $\phi_j \to 0$ in $\mathscr{S}$, then we've just established that $\phi_j \to 0$ in $\mathscr{E}(\mathbb{R}^n)$. As a consequence, we have that $u[\phi_j] \to 0$ by the continuity of $u : \mathscr{E}(\mathbb{R}^n) \to \mathbb{C}$. Thus $u : \mathscr{S} \to \mathbb{C}$ is continuous, so we have shown that $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. This establishes the result.

Exercise 3.3. Let $\{a_j\}_{j=1}^\infty \subset \mathbb{R}$ be a sequence of real numbers. Define for $\phi \in C^\infty(\mathbb{R})$:

$$u[\phi] = \sum_{j=1}^\infty a_j \phi(j)$$

provided that the sum converges. Give necessary and sufficient conditions on a_j such that:

a) $u \in \mathscr{E}'(\mathbb{R})$.

Solution There must exist J such that $a_j = 0$ for all $j > J$. This is because $\operatorname{supp} u = \{j \in \mathbb{N} : a_j \neq 0\}$. Thus u has compact support if and only if $\{a_j\}$ is eventually zero.

b) $u \in \mathscr{S}'$.

Solution By Lemma 2.11, $u \in \mathscr{S}'$ if and only if there exist C, N such that:

$$|a_j| \leq C j^N.$$

c) $u \in \mathscr{D}'(\mathbb{R})$.

Solution u is always in $\mathscr{D}'(\mathbb{R}^n)$, since a compactly supported function will only ever see finitely many of the terms in the sequence, so there is no issue of convergence at infinity.

Exercise 3.4. For $\xi \in \mathbb{R}^n$, define $e_\xi(x) = e^{i\xi\cdot x}$. Show that $T_{e_\xi} \in \mathscr{S}'$, and that:

$$T_{e_\xi} \to 0, \qquad \text{as } |\xi| \to \infty$$

in the topology[2] of $\mathscr{S}'$.

Solution Suppose $\phi \in \mathscr{S}$. We note that:

$$\begin{aligned} ||\phi||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\phi(x)|\, dx &= \int_{\mathbb{R}^n} (1+|x|)^{n+1} |\phi(x)| \times (1+|x|)^{-n-1} dx \\ &\leq \sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |\phi(x)| \times \int_{\mathbb{R}^n} (1+|y|)^{-n-1} dy \\ &\leq C \sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |\phi(x)| \end{aligned}$$

[2] This is defined precisely as the topology of $\mathscr{D}'(\Omega)$, *mutatis mutandis*.

Thus necessarily $\phi \in L^1$, so since $|e_\xi| = 1$, we have $e_\xi(x)\phi(x) \in L^1(\mathbb{R}^n)$ and so the map:

$$T_{e_\xi}[\phi] = \int_{\mathbb{R}^n} e^{i\xi\cdot x}\phi(x)dx$$

is well defined and linear. We also note that:

$$\left|T_{e_\xi}[\phi]\right| \leq \int_{\mathbb{R}^n} \left|e^{i\xi\cdot x}\phi(x)\right| dx = ||\phi||_{L^1(\mathbb{R}^n)} \leq C \sup_{x\in\mathbb{R}^n} (1+|x|)^{n+1} |\phi(x)|,$$

so T_{e_ξ} is continuous by Lemma 2.11. Thus $T_{e_\xi} \in \mathscr{S}'$.

Let $\phi \in \mathscr{S}$. Then:

$$T_{e_\xi}[\phi] = \int_{\mathbb{R}^n} e^{i\xi\cdot x}\phi(x)dx = \hat{\phi}(-\xi).$$

By Lemma 3.1, we have that $T_{e_\xi}[\phi] \to 0$ as $\xi \to 0$. Since $\phi \in \mathscr{S}$ was arbitrary, we conclude that $T_{e_\xi} \to 0$ in $\mathscr{S}'$.

Exercise 3.5. Calculate the Fourier transform of the following functions $f \in L^1(\mathbb{R})$:

a) $f(x) = \dfrac{\sin x}{1+x^2}$.

Solution We know that if:

$$g(x) = \frac{1}{1+x^2},$$

then:

$$\hat{g}(\xi) = \begin{cases} \pi e^{\xi} & \xi < 0, \\ \pi e^{-\xi} & \xi \geq 0. \end{cases}$$

Note that:

$$f = \frac{1}{2i}\left(e_x g - e_{-x} g\right)$$

so applying Lemma 3.2, we deduce:

$$\hat{f}(\xi) = \frac{1}{2i}\left(\widehat{e_1 g}(\xi) - \widehat{e_{-1} g}(\xi)\right) = \frac{1}{2i}\left(\widehat{g}(\xi-1) - \widehat{g}(\xi+1)\right)$$

So that:

$$\hat{g}(\xi) = \begin{cases} \frac{\pi}{2i}(e^{\xi-1} - e^{\xi+1}) & \xi < -1, \\ \frac{\pi}{2i}(e^{\xi-1} - e^{-\xi-1}) & -1 \leq \xi \leq 1, \\ \frac{\pi}{2i}(e^{-\xi+1} - e^{-\xi-1}) & \xi > 1, \end{cases}$$

which we can tidy up to give:

$$\hat{g}(\xi) = \begin{cases} i\pi e^{\xi} \sinh 1 & \xi < -1, \\ -i\pi e^{-1} \sinh \xi & -1 \leq \xi \leq 1, \\ -i\pi e^{-\xi} \sinh 1 & \xi > 1, \end{cases}$$

b) $f(x) = \dfrac{1}{\epsilon^2 + x^2}$, for $\epsilon > 0$ a constant.

Solution We know that if:

$$g(x) = \frac{1}{1+x^2},$$

then:

$$\hat{g}(\xi) = \begin{cases} \pi e^{\xi} & \xi < 0, \\ \pi e^{-\xi} & \xi \geq 0. \end{cases}$$

Note that:

$$f(x) = \epsilon^{-2} g(\epsilon^{-1} x) = \epsilon^{-1} g_\epsilon(x)$$

so applying Lemma 3.2, we deduce:

$$\hat{f}(\xi) = \epsilon^{-1} \hat{g}(\epsilon x)$$

so that:

$$\hat{f}(\xi) = \begin{cases} \frac{\pi}{\epsilon} e^{\epsilon\xi} & \xi < 0, \\ \frac{\pi}{\epsilon} e^{-\epsilon\xi} & \xi \geq 0. \end{cases}$$

c) $f(x) = \sqrt{\dfrac{\sigma}{t}} e^{-\sigma\frac{(x-y)^2}{t}}$, where $\sigma > 0$, $t > 0$ and y are constants.

Solution We know that if $g(x) = e^{-\frac{1}{2}x^2}$, then $g(\xi) = \sqrt{2\pi} e^{-\frac{1}{2}\xi^2}$. We note that:

$$f = \frac{1}{\sqrt{2}} \tau_y (g_\kappa)$$

where $\kappa = \sqrt{\frac{t}{2\sigma}}$. Thus we find:

$$\hat{f} = \sqrt{\pi} e^{-\frac{t}{4\sigma}\xi^2} e^{-iy\xi}.$$

*d) $f(x) = \dfrac{1}{\cosh x}$.

Solution We need to evaluate the integral:

$$\hat{f}(\xi) = \int_{\mathbb{R}} \frac{e^{-ix\xi}}{\cosh x} dx$$

Consider the contour $\Gamma^{(R)} = \Gamma_1^{(R)} \cup \Gamma_2^{(R)} \cup \Gamma_3^{(R)} \cup \Gamma_4^{(R)}$, where:

$$\begin{aligned} \Gamma_1^{(R)}(t) &= t, & -R \leq t \leq R \\ \Gamma_2^{(R)}(t) &= R + it, & 0 \leq t \leq \pi \\ \Gamma_3^{(R)}(t) &= -t + \pi i, & -R \leq t \leq R \\ \Gamma_4^{(R)}(t) &= -R + i(\pi - t) & 0 \leq t \leq \pi \end{aligned}$$

And let us look at:

$$I = \oint_{\Gamma^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz = \oint_{\Gamma^{(R)}} g(z) dz$$

From the expression:

$$\cosh(x+iy) = \cos y \cosh x + i \sin y \sinh x$$

we know that $\cosh(z)$ vanishes for $z = \frac{1}{2}(2n+1)\pi i$, where $n \in \mathbb{Z}$. Thus to evaluate I using Cauchy's formula, we have to consider the residue of g at $z = i\pi/2$. We find (using that if $g(z) = h(z)/k(z)$ and $k(z)$ has a simple zero at c then $Res(g;x) = h(c)/k'(c)$):

$$Res\left(g; \frac{i\pi}{2}\right) = \frac{e^{\frac{\pi}{2}\xi}}{\sinh \frac{i\pi}{2}} = -ie^{\frac{\pi}{2}\xi}$$

Thus,

$$I = 2\pi e^{\frac{\pi}{2}\xi}.$$

Now let us consider the contributions from the various part of the contour. On $\Gamma_2^{(R)}(t)$ we have the estimate:

$$\begin{aligned}|\cosh(R+it)|^2 &= \cos^2 t \cosh^2 R + \sin^2 t \sinh^2 R \\ &\geq \cos^2 t \sinh^2 R + \sin^2 t \sinh^2 R = \sinh^2 R\end{aligned}$$

so that:

$$\begin{aligned}\left| \int_{\Gamma_2^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz \right| &= \left| \int_0^\pi \frac{e^{(-iR+t\xi)}}{\cosh(R+i)t} dt \right| \\ &\leq \int_0^\pi \left| \frac{e^{(-iR+t\xi)}}{\cosh(R+it)} \right| dt \\ &\leq \frac{\pi e^{\xi\pi}}{\sinh R} \to 0\end{aligned}$$

as $R \to \infty$. A similar estimate shows that:

$$\left| \int_{\Gamma_4^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz \right| \to 0,$$

as $R \to \infty$. We also note that:

$$\int_{\Gamma_1^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz = \int_{-R}^{R} \frac{e^{-it\xi}}{\cosh t} dt \to \hat{f}(\xi)$$

as $\xi \to \infty$, while:

$$\int_{\Gamma_4^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz = -\int_{-R}^{R} \frac{e^{it\xi+\pi\xi}}{\cosh t} dt = e^{\pi\xi} \int_{-R}^{R} \frac{e^{-it\xi}}{\cosh t} dt \to e^{\pi\xi} \hat{f}(\xi)$$

as $R \to 0$. Now, using that:

$$2\pi e^{\frac{\pi}{2}\xi} = I = \int_{\Gamma_1^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz + \int_{\Gamma_2^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz + \int_{\Gamma_3^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz + \int_{\Gamma_4^{(R)}} \frac{e^{-iz\xi}}{\cosh z} dz.$$

we have, after sending $R \to \infty$:

$$2\pi e^{\frac{\pi}{2}\xi} = (1 + e^{\pi\xi})\hat{f}(\xi),$$

so that

$$\hat{f}(\xi) = \frac{\pi}{\cosh\left(\frac{\pi\xi}{2}\right)}.$$

Exercise 3.6. Suppose $f \in C^1(\mathbb{R}^n)$ and that $f, D_j f \in L^1(\mathbb{R}^n)$. Fix $\epsilon > 0$. Show that there exists $f_\epsilon \in C_0^1(\mathbb{R}^n)$ such that

$$||f - f_\epsilon||_{L^1(\mathbb{R}^n)} + ||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

[Hint: First construct, for large R, a smooth cut-off function $\chi_R(x)$ with $\chi_R(x) = 1$ for $|x| < R$, $\chi_R(x) = 0$ for $|x| > 2R$ and $|D\chi_R(x)| < C$, where C is independent of R.]

Solution Let $\chi \in C_c^\infty(B_2(0))$ be a smooth function which satisfies $0 \leq \chi \leq 1$, with $\chi(x) = 1$ for $x \in B_1(0)$. Define:

$$\chi_R(x) = \chi\left(\frac{x}{R}\right).$$

We have $\chi_R(x) = 1$ for $|x| < R$, $\chi_R(x) = 0$ for $|x| > 2R$ and

$$D\chi_R(x) = \frac{1}{R} D\chi\left(\frac{x}{R}\right),$$

so that:

$$\sup_{x \in \mathbb{R}^n} |D\chi_R(x)| \leq \frac{1}{R} \sup_{D\chi(x)} \leq C,$$

where C is independent of R. Now, since $f, D_j f \in L^1(\mathbb{R}^n)$, given $\delta > 0$, there exists K such that:

$$\int_{\mathbb{R}^n \setminus B_K(0)} |f(x)|\, dx \leq \delta, \qquad \int_{\mathbb{R}^n \setminus B_K(0)} |D_j f(x)|\, dx \leq \delta.$$

We take $R_\delta > 2K + 1$, and we estimate:

$$\begin{aligned}
||f - \chi_{R_\delta} f||_{L^1(\mathbb{R}^n)} &= \int_{B_{2R_\delta}(0) \setminus B_{R_\delta}(0)} |f(x)|\, (1 - \chi_{R_\delta}(x)) dx \\
&\leq \int_{B_{2R_\delta}(0) \setminus B_{R_\delta}(0)} |f(x)|\, dx \leq \int_{\mathbb{R}^n \setminus B_K(0)} |f(x)|\, dx \\
&< \delta
\end{aligned}$$

Similarly, we estimate:

$$\begin{aligned}||D_j f - D_j(\chi_{R_\delta} f)||_{L^1(\mathbb{R}^n)} &= \int_{B_{2R_\delta}(0)\backslash B_{R_\delta}(0)} |D_j f(x)|\,(1-\chi_{R_\delta}(x))dx \\ &\quad + \int_{B_{2R_\delta}(0)\backslash B_{R_\delta}(0)} |f(x)|\,|D\chi_{R_\delta}(x)|\,dx \\ &\leq (1+C)\int_{B_{2R_\delta}(0)\backslash B_{R_\delta}(0)} |f(x)|\,dx \\ &\leq (1+C)\int_{\mathbb{R}^n\backslash B_K(0)} |f(x)|\,dx \\ &< (1+C)\delta\end{aligned}$$

Taking δ sufficiently small that $(2+C)\delta < \epsilon/2$ and setting $f_\epsilon = \chi_{R_\delta} f$, we have shown:

$$||f - f_\epsilon||_{L^1(\mathbb{R}^n)} + ||D_j f - D_j f_\epsilon||_{L^1(\mathbb{R}^n)} < \frac{\epsilon}{2}.$$

Exercise 3.7. Suppose $f \in L^1(\mathbb{R}^n)$, with $\operatorname{supp} f \subset B_R(0)$ for some $R > 0$.

a) Show that $\hat{f} \in C^\infty(\mathbb{R}^n)$ and for any multi-index:

$$\sup_{\xi\in\mathbb{R}^n} \left|D^\alpha \hat{f}(\xi)\right| \leq R^{|\alpha|}\,||f||_{L^1(\mathbb{R}^n)}$$

Solution Recalling that $D^\alpha \hat{f}(\xi) = (-i)^{|\alpha|}\widehat{x^\alpha f}(\xi)$, we can estimate:

$$\begin{aligned}\left|D^\alpha \hat{f}(\xi)\right| = \left|\widehat{x^\alpha f}(\xi)\right| &= \left|\int_{\mathbb{R}^n} x^\alpha f(x) e^{-i\xi\cdot x} dx\right| \\ &= \left|\int_{B_R(0)} x^\alpha f(x) e^{-i\xi\cdot x} dx\right| \\ &\leq \int_{B_R(0)} \left|x^\alpha f(x) e^{-i\xi\cdot x}\right| dx \\ &\leq R^{|\alpha|}\int_{B_R(0)} |f(x)|\,dx = R^{|\alpha|}\,||f||_{L^1(\mathbb{R}^n)}\,.\end{aligned}$$

Here we have used the compactness of the support of f, and the fact that $\left|x^i\right| < R$ if $x \in B_R(0)$. Taking the supremum over ξ, the result follows.

b) Show that $\hat{f}$ is real analytic, with an infinite radius of convergence, i.e.:

$$\hat{f}(\xi) = \sum_\alpha D^\alpha \hat{f}(0)\frac{\xi^\alpha}{\alpha!}$$

holds for all $\xi \in \mathbb{R}^n$.

Solution We know that $\hat{f}$ is smooth on $\mathbb{R}^n$, so fixing some $r > 0$, we can apply Taylor's theorem to find that for any $\xi \in B_r(0)$ and any $k \in \mathbb{N}$ we have:

$$\hat{f}(\xi) = \sum_{|\alpha| \leq k} D^\alpha \hat{f}(0) \frac{\xi^\alpha}{\alpha!} + \sum_{|\beta|=k+1} R_\beta(\xi) x^\beta$$

where:

$$|R_\beta(\xi)| \leq \frac{1}{\beta!} \max_{|\alpha|=|\beta|} \max_{\eta \in \overline{B_r(0)}} \left| D^\alpha \hat{f}(\eta) \right|.$$

In order to show that the Taylor series converges as $k \to \infty$, we need to show that

$$\left| \sum_{|\beta|=k+1} R_\beta(\xi) x^\beta \right| \to 0,$$

as $k \to \infty$. We estimate:

$$\left| \sum_{|\beta|=k+1} R_\beta(\xi) x^\beta \right| \leq \# \{\beta : |\beta| = k+1\} \times \sup_{|\beta|=k+1} |R_\beta(\xi)| \times r^{k+1}$$

Note that by part a), we already have that:

$$\sup_{|\beta|=k+1} |R_\beta(\xi)| \leq R^{k+1} ||f||_{L^1(\mathbb{R}^n)} \sup_{|\beta|=k+1} \frac{1}{\beta!}.$$

Recalling that $\beta! = \beta_1! \beta_2! \cdots \beta_n!$, we can see that $\beta! \geq \lfloor \frac{k+1}{n} \rfloor !$, so that:

$$\sup_{|\beta|=k+1} |R_\beta(\xi)| \leq \frac{R^{k+1}}{\lfloor \frac{k+1}{n} \rfloor !} ||f||_{L^1(\mathbb{R}^n)} \leq R^{k+1} \left(\frac{\lfloor \frac{k+1}{n} \rfloor}{e} \right)^{-\lfloor \frac{k+1}{n} \rfloor} ||f||_{L^1(\mathbb{R}^n)}$$

where we've used the result (Stirling's approximation) that:

$$p! \geq \left(\frac{p}{e} \right)^p.$$

Now we need to estimate:

$$\# \{\beta : |\beta| = k+1\}.$$

This is the number of ways of partitioning $k+1$ into at most n integers, or equivalently the number of ways of putting $k+1$ identical objects into n boxes. We can crudely overestimate this by assuming that all of the objects are distinguishable, in which case there are n^{k+1} ways to assign the objects to their boxes. Thus we have:

$$\# \{\beta : |\beta| = k+1\} \leq n^{k+1}.$$

Finally then, we have:

$$\left| \sum_{|\beta|=k+1} R_\beta(\xi) x^\beta \right| \leq (nrR)^{k+1} \left(\frac{\lfloor \frac{k+1}{n} \rfloor}{e} \right)^{-\lfloor \frac{k+1}{n} \rfloor} ||f||_{L^1(\mathbb{R}^n)}$$

Now, suppose that k is sufficiently large that $\left(\frac{\lfloor \frac{k+1}{n} \rfloor}{e}\right)^{\frac{1}{n}} \geq (2nrR)$. Then:

$$(nrR)^{k+1}\left(\frac{\lfloor \frac{k+1}{n} \rfloor}{e}\right)^{-\lfloor \frac{k+1}{n} \rfloor} \leq \left[\left(\frac{\lfloor \frac{k+1}{n} \rfloor}{Rrne}\right)^{\frac{1}{n}}\right]^{-(k+1)} \leq 2^{-k},$$

so that indeed we have:

$$\left|\sum_{|\beta|=k+1} R_\beta(\xi)x^\beta\right| \to 0,$$

as $k \to \infty$. Thus the Taylor series converges for $\xi \in B_r(0$. Since r was arbitrary, we conclude that the Taylor series converges everywhere.

c) Show that if $\hat{f}(\xi)$ vanishes on an open set, it must vanish everywhere.
[Hint: use part i) of Lemma 3.2]

Solution First suppose that $\hat{f}(\xi)$ vanishes in a neighbourhood of the origin. Then $D^\alpha \hat{f}(0) = 0$ for all α, and so by the previous part, we have that $\hat{f} = 0$. We conclude that if a compactly supported function has a Fourier transform which vanishes in the neighbourhood of the origin, then the function (and its transform) must vanish everyhwhere.

Now, suppose that $\hat{f}(\xi)$ is the Fourier transform of a compactly supported function and that $\hat{f}(\xi)$ vanishes in a neighbourhood of η. Then $\tau_{-\eta}\hat{f}(\xi)$ vanishes in a neighbourhood or the origin and moreover:

$$\tau_{-\eta}\hat{f}(\xi) = \widehat{e_{-\eta}f}(\xi),$$

so that $\tau_{-\eta}\hat{f}(\xi)$ is the transform of $e_{-\eta}f$, which has compact support because f does. Thus $\tau_{-\eta}\hat{f}(\xi)$ is the Fourier transform of a compactly supported function and $\tau_{-\eta}\hat{f}(\xi)$ vanishes in a neighbourhood of the origin. We deduce that $\tau_{-\eta}\hat{f}(\xi)$ and hence $\hat{f}(\xi)$ vanishes everywhere.

You may assume the following form of Taylor's theorem. Suppose $g \in C^{k+1}(\overline{B_r(0)})$. Then for $x \in B_r(0)$:

$$g(x) = \sum_{|\alpha|\leq k} D^\alpha g(0)\frac{\xi^\alpha}{\alpha!} + \sum_{|\beta|=k+1} R_\beta(x)x^\beta$$

where the remainder $R_\beta(x)$ satisfies the following estimate in $B_r(0)$:

$$|R_\beta(x)| \leq \frac{1}{\beta!}\max_{|\alpha|=|\beta|}\max_{y\in\overline{B_r(0)}}|D^\alpha g(y)|.$$

See §A.1 of the notes for notation.

Exercise 4.1. Consider the following ODE problem. Given $f : \mathbb{R} \to \mathbb{C}$, find ϕ such that:

$$-\phi'' + \phi = f. \tag{2}$$

a) Show that if $f \in \mathscr{S}$, there is a unique $\phi \in \mathscr{S}$ solving (1), and give an expression for $\hat{\phi}$.

b) Show that

$$\phi(x) = \int_{\mathbb{R}} f(y)G(x-y)dy$$

where

$$G(x) = \begin{cases} \frac{1}{2}e^{x} & x < 0, \\ \frac{1}{2}e^{-x} & x \geq 0. \end{cases}$$

Exercise 4.2. Suppose $f \in L^1(\mathbb{R}^3)$ is a radial function, i.e. $f(Rx) = f(x)$, whenever $R \in SO(3)$ is a rotation.

a) Show that $\hat{f}$ is radial.

b) Suppose that $\xi = (0, 0, \zeta)$. By writing the Fourier integral in polar coordinates, show that

$$\hat{f}(\xi) = \int_{r=0}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f(r)e^{-i\zeta r \cos\theta} r^2 \sin\theta d\theta dr d\phi.$$

c) Making the substitution $s = \cos\theta$, and using the fact that $\hat{f}$ is radial, deduce:

$$\hat{f}(\xi) = 4\pi \int_0^{\infty} f(r) \frac{\sin r\,|\xi|}{r\,|\xi|} r^2 dr$$

for any $\xi \in \mathbb{R}^n$.

Exercise 4.3. (*) Suppose that $f, g \in L^2(\mathbb{R}^n)$, and denote the Fourier-Plancherel transform by $\overline{\mathcal{F}}$. You may assume any results already established for the Fourier transform.

a) Show that

$$(f, g) = \frac{1}{(2\pi)^n} \left(\overline{\mathcal{F}}[g], \overline{\mathcal{F}}[g]\right).$$

b) Recall that $\check{f}(y) = f(-y)$. Show that:

$$\overline{\mathcal{F}}\left[\overline{\mathcal{F}}[f]\right] = (2\pi)^n \check{f}.$$

Hence, or otherwise, deduce that $\overline{\mathcal{F}} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bijection, and that $\overline{\mathcal{F}}^{-1} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a bounded linear map.

c) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \lim_{R\to\infty} \int_{B_R(0)} f(x)e^{-ix\cdot\xi}dx$$

with convergence in the sense of $L^2(\mathbb{R}^n)$.

d) Suppose that $f \in C^1(\mathbb{R}^n)$ and $f, D_j f \in L^2(\mathbb{R}^n)$. Show that $\xi_j\overline{\mathcal{F}}[f](\xi) \in L^2(\mathbb{R}^n)$ and:

$$\overline{\mathcal{F}}[D_j f](\xi) = i\xi_j\overline{\mathcal{F}}[f](\xi)$$

e) For $x \in \mathbb{R}$ let:

$$f(x) = \frac{\sin x}{x}$$

i) Show that $f \in L^2(\mathbb{R})$.

ii) Show that:

$$\overline{\mathcal{F}}[f](\xi) = \begin{cases} \pi & -1 < \xi < 1, \\ 0 & |\xi| \geq 1. \end{cases}$$

f) i) Show that for all $x \in \mathbb{R}^n$:

$$|f \star g(x)| \leq ||f||_{L^2(\mathbb{R}^n)} ||g||_{L^2(\mathbb{R}^n)} .$$

ii) Show that $f \star g \in C^0(\mathbb{R}^n)$ and:

$$f \star g = \mathcal{F}^{-1}\left[\overline{\mathcal{F}}[f] \cdot \overline{\mathcal{F}}[g]\right]$$

where:

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x}d\xi.$$

[Hint for parts a), b), d), f): approximate by Schwartz functions]

Exercise 4.4. Work in $\mathbb{R}^3$. For $k > 0$, define the function:

$$G(x) = \frac{e^{-k|x|}}{4\pi |x|}$$

a) Show that $G \in L^1(\mathbb{R}^3)$.

b) Show that:

$$\hat{G}(\xi) = \frac{1}{|\xi|^2 + k^2}$$

[Hint: use Exercise 4.2, part c)]

Exercise 4.5. Consider the inhomogeneous Helmholtz equation on $\mathbb{R}^3$:

$$-\Delta\phi + k^2\phi = f \tag{3}$$

where $f \in \mathscr{S}$. Show that there exists a unique $\phi \in \mathscr{S}$ satisfying (2) given by:

$$\phi(x) = \int_{\mathbb{R}^3} f(y)G(x-y)dy,$$

where

$$G(x) = \frac{e^{-k|x|}}{4\pi\,|x|}.$$

[Hint: first derive an equation satisfied by $\hat{\phi}$]

Exercise 4.6. Verify that if $f \in L^1_{loc.}$ is such that $T_f \in \mathscr{S}'$, then:

$$\tau_x T_f = T_{\tau_x f}, \quad \text{and} \quad \check{T}_f = T_{\check{f}}$$

Exercise 4.7. Let $f : \mathbb{R} \to \mathbb{R}$ be the sign function

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

and define $f_R(x) = f(x)\mathbb{1}_{[-R,R]}(x)$.

a) Sketch $f_R(x)$.

b) Show that:

$$T_{f_R} \to T_f \text{ in } \mathscr{S}' \text{ as } R \to \infty.$$

c) Show that:

$$\hat{f}_R(\xi) = 2i\frac{\cos R\xi - 1}{\xi}$$

d) For $\phi \in \mathscr{S}$, show that:

$$T_{\hat{f}_R}[\phi] = -2i\int_0^\infty \frac{\phi(x)-\phi(-x)}{x}dx + 2i\int_0^\infty \left(\frac{\phi(x)-\phi(-x)}{x}\right)\cos Rx dx$$

e) By applying the Riemann-Lebesgue Lemma, or otherwise, show that for any $\psi \in \mathscr{S}$:

$$\int_0^\infty \psi(x)\cos Rx dx \to 0$$

as $R \to \infty$.

f) Deduce that

$$\widehat{T_f} = -2iP.V.\left(\frac{1}{x}\right)$$

g) Write down $\widehat{T_H}$, where H is the Heaviside function:

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Exercise 5.1. Suppose $v \in \mathscr{E}'(\mathbb{R}^n)$ and let:

$$u = \sum_{g \in \mathbb{Z}^n} \tau_g v.$$

Show that if $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset K$ for some compact $K \subset \mathbb{R}^n$ then

$$u[\phi] = \sum_{g \in A} \tau_g v[\phi],$$

for some finite set $A \subset \mathbb{Z}^n$ which depends only on K. Deduce that u defines a distribution.

Solution Suppose $\phi \in \mathscr{D}(\mathbb{R}^n)$ with $\operatorname{supp} v \subset K$, with K some fixed, compact K. Pick R large enough that $\operatorname{supp} v \subset B_R(0)$ and $K \subset B_R(0)$. Suppose now that $|g| > 2R$ and that $x \in K - g$, so that $x = y - g$ for some $y \in K \subset B_R(0)$. From the triangle inequality we have:

$$|x| \geq |g| - |y| > R$$

so that $(K - g) \cap B_R(0) = \emptyset$. Now consider $\tau_g v[\phi] = v[\tau_{-g}\phi]$. We have that $\operatorname{supp} \tau_{-g}\phi = (\operatorname{supp} \phi - g) \subset (K - g)$ and $\operatorname{supp} v \subset B_R(0)$, so that $\operatorname{supp} \tau_{-g}\phi \cap \operatorname{supp} v = \emptyset$. Thus $\tau_g v[\phi] = 0$. Thus:

$$u[\phi] = \sum_{g \in \mathbb{Z}^n} \tau_g v[\phi] = \sum_{g \in \mathbb{Z}^n \cap B_{2R}(0)} \tau_g v[\phi].$$

so that the sum is in fact finite.

Now consider a sequence $\{\phi_j\}_{j=1}^{\infty} \subset \mathscr{D}(\mathbb{R}^n)$ with $\phi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$. We have that $\operatorname{supp} \phi_j \subset K$ for some fixed compact set. Applying the previous result, there exists $R > 0$ depending on K such that:

$$u[\phi_j] = \sum_{g \in \mathbb{Z}^n \cap B_{2R}(0)} \tau_g v[\phi_j]$$

This is a finite sum, and since $v \in \mathscr{E}'(\mathbb{R}^n)$, we have that each term $\tau_g v[\phi_j] \to 0$ as $j \to \infty$, so that:

$$u[\phi_j] \to 0,$$

and $u \in \mathscr{D}'(\mathbb{R}^n)$.

Exercise 5.2. Recall that for $x \in \mathbb{R}^n$:

$$||x||_1 := \sum_{i=1}^{n} |x_i|.$$

For $k \in \mathbb{N}$ set:

$$Q_k = \left\{ g \in \mathbb{Z}^n : k - \frac{1}{2} \leq ||g||_1 < k + \frac{1}{2} \right\}$$

2

a) Show that:

$$\#Q_k = (2k+1)^n - (2k-1)^n$$

so that $\#Q_k \leq c(1+k)^{n-1}$ for some $c > 0$.

Solution Let C_k be the lattice cube:

$$\left\{ g \in \mathbb{Z}^n : ||g||_1 < k + \frac{1}{2} \right\}$$

This has $2k+1$ lattice points on each side, so $\#C_k = (2k+1)^n$. Clearly $Q_k = C_k \setminus C_{k-1}$ and the result follows. To see the bound for $\#Q_k$, notice that it is a polynomial of degree $n-1$ whose coefficients depend only on n, so it must be bounded by $c(1+k)^{n-1}$ for some c sufficiently large.

b) By applying the Cauchy-Schwartz identity to estimate $a \cdot b$ for $a = (1, \dots, 1)$ and $b = (|g_1|, \dots, |g_n|)$, deduce that:

$$||g||_1 \leq \sqrt{n}\, |g|$$

Solution The Cauchy–Schwartz inequality tells us that:

$$a \cdot b \leq |a|\, |b|$$

Applying this with a, b as suggested, we have:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = \sum_{i=1}^{n} |g_i| = ||g||_1 .$$

On the other hand:

$$|a|^2 = \sum_{i=1}^{n} |a_i|^2 = \sum_{i=1}^{n} 1 = n$$

and

$$|b|^2 = \sum_{i=1}^{n} |b_i|^2 = \sum_{i=1}^{n} |g_i|^2 = |g|^2 ,$$

and the result follows.

c) Show that there exists a constant $C > 0$, depending only on n such that:

$$\sum_{g \in \mathbb{Z}^n; ||g||_1 \leq K} \frac{1}{(1+|g|)^{n+1}} \leq 1 + C \sum_{k=1}^{K} \frac{1}{k^2}$$

holds for all $K \in \mathbb{N}$. Deduce that:

$$\sum_{g \in \mathbb{Z}^n} \frac{1}{(1+|g|)^{n+1}} < \infty.$$

Solution We note that:

$$\frac{1}{(1+|g|)^{n+1}} \leq \frac{1}{(1+n^{-\frac{1}{2}}\,||g||_1)^{n+1}} \leq \frac{n^{\frac{n+1}{2}}}{(n^{\frac{1}{2}}+||g||_1)^{n+1}} \leq \frac{n^{\frac{n+1}{2}}}{(1+||g||_1)^{n+1}}.$$

We can split up the sum as:

$$\sum_{g\in\mathbb{Z}^n;||g||_1\leq K} \frac{1}{(1+|g|)^{n+1}} = 1 + \sum_{k=1}^{K}\sum_{g\in Q_k} \frac{1}{(1+|g|)^{n+1}}$$

and we estimate the second term by:

$$\begin{aligned}\sum_{k=1}^{K}\sum_{g\in Q_k} \frac{1}{(1+|g|)^{n+1}} &\leq n^{\frac{n+1}{2}} \sum_{k=1}^{K}\sum_{g\in Q_k} \frac{1}{(1+||g||_1)^{n+1}} \\ &= n^{\frac{n+1}{2}} \sum_{k=1}^{K} \frac{\#Q_k}{(1+k)^{n+1}} \\ &\leq n^{\frac{n+1}{2}} c \sum_{k=1}^{K} \frac{1}{(1+k)^2} \leq C\sum_{k=1}^{K}\frac{1}{k^2}\end{aligned}$$

Now, this sum converges to a finite value as $K\to\infty$, so we have that

$$\sum_{g\in\mathbb{Z}^n;||g||_1\leq K} \frac{1}{(1+|g|)^{n+1}}$$

is increasing and bounded above, hence converges to a finite value.

Exercise 5.3. Show that if c_g satisfy:

$$|c_g| \leq K(1+|g|)^N$$

for some $K>0$ and $N\in\mathbb{N}$, then:

$$\sum_{g\in\mathbb{Z}^n} c_g\delta_{2\pi g}$$

converges in $\mathscr{S}'$.

Solution Fix any $M\geq 0$. First recall that if $\phi\in\mathscr{S}$, then:

$$\sup_{x\in\mathbb{R}^n}(1+|x|)^M\,|\psi(x)| < \infty$$

We deduce that:

$$|\psi(2\pi g)| \leq \frac{C}{(1+|g|)^M}$$

holds for some C depending on ψ, M. Letting $M=N+n+1$, we have:

$$\sum_{g\in\mathbb{Z}^n,|g|<k} |c_g\delta_{2\pi g}[\phi]| \leq \sum_{g\in\mathbb{Z}^n,|g|<k} |c_g|\,|\phi(2\pi g)| \leq CK\sum_{g\in\mathbb{Z}^n,|g|<k}\frac{1}{(1+|g|)^{n+1}}$$

The sum on the right converges as $k \to \infty$, so we conclude that:

$$\sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}[\phi]$$

converges absolutely for any $\phi \in \mathscr{S}$. This is precisely the statement that

$$\sum_{g \in \mathbb{Z}^n} c_g \delta_{2\pi g}$$

converges in $\mathscr{S}'$.

Exercise 5.4. Suppose $f \in L^p_{loc.}(\mathbb{R}^n)$ is a periodic function. Fix $\epsilon > 0$, and let:

$$Q = \{x \in \mathbb{R}^n : |x_j| < 1, j = 1, \ldots, n\}, \qquad q = \left\{x \in \mathbb{R}^n : |x_j| < \frac{1}{2}, j = 1, \ldots, n\right\}$$

a) Show that there exists $h_\epsilon \in C^\infty(\mathbb{R}^n)$ with:

$$\operatorname{supp} h_\epsilon \subset Q$$

such that:

$$||f \mathbb{1}_q - h_\epsilon||_{L^p(\mathbb{R}^n)} < \epsilon.$$

Solution We have that $f\mathbb{1}_q \in L^p(\mathbb{R}^n)$. Applying Theorem 1.13, we can construct $h_\epsilon \in C^\infty$ such that $h_\epsilon \to f\mathbb{1}_q$ in $L^p(\mathbb{R}^n)$, and with $\operatorname{supp} h_\epsilon \subset \operatorname{supp} f\mathbb{1}_q + B_\epsilon(0) \subset Q$.

Define

$$f_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_g h_\epsilon$$

b) Show that f_ϵ is smooth and periodic.

Solution Since h_ϵ has support inside Q, the sum is locally finite. That is to say that if U is any bounded open set,

$$f_\epsilon|_U = \sum_{g \in \mathbb{Z}^n} \tau_g h_\epsilon = \sum_{g \in \mathbb{Z}^n \cap E} \tau_g h_\epsilon$$

for some bounded set E, so that f_ϵ is certainly smooth in U since it is a finite sum of smooth functions. To see that f_ϵ is periodic, we note that if $g' \in \mathbb{Z}^n$, then:

$$\tau_{g'} f_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_{g'} \tau_g h_\epsilon = \sum_{g \in \mathbb{Z}^n} \tau_{g+g'} h_\epsilon = \sum_{g'' \in \mathbb{Z}^n} \tau_{g''} h_\epsilon = f_\epsilon$$

c) Show that there exists a constant c_n depending only on n such that:

$$||f - f_\epsilon||_{L^p(q)} < c_n \epsilon.$$

Solution Noting that:

$$||f\mathbb{1}_q - h_\epsilon||_{L^p(\mathbb{R}^n)} = ||f - h_\epsilon||_{L^p(q)} + ||h_\epsilon||_{L^p(Q\setminus q)}$$

we deduce that:

$$||f - h_\epsilon||_{L^p(q)} + ||h_\epsilon||_{L^p(Q\setminus q)} < \epsilon.$$

Now note that:

$$(f - f_\epsilon)|_q = f - h_\epsilon - \sum_{|g|\leq n} \tau_g h_\epsilon$$

since if $g \in \mathbb{Z}^n$ with $|g| > n$ and $x \in q$, then $x - g \notin Q$. We deduce:

$$||f - f_\epsilon||_{L^p(q)} \leq \epsilon + \sum_{|g|=1} \epsilon = c_n \epsilon.$$

Exercise 5.5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = x \quad \text{for} \quad |x| < \frac{1}{2}, \qquad f(x+1) = f(x).$$

Show that:

$$f(x) = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{i(-1)^n}{2\pi n} e^{2\pi i n x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n x),$$

with convergence in $L^2_{loc.}(\mathbb{R})$.

Solution By Theorem 3.17 and Corollary 3.18 we know that for $f \in L^2_{loc.}(\mathbb{R})$ a periodic function, we have that:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x}$$

where the sum converges in $L^2_{loc.}$ and the Fourier coefficients are given by:

$$f_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} f(x) dx.$$

We calculate for $n \neq 0$.

$$\begin{aligned} f_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i n x} f(x) dx &= \int_{-\frac{1}{2}}^{\frac{1}{2}} x e^{-2\pi i n x} dx \\ &= \left[x \frac{e^{-2\pi i n x}}{-2\pi n i} \right]_{-\frac{1}{2}}^{\frac{1}{2}} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-2\pi i n x}}{-2\pi n i} dx \\ &= \frac{i}{2\pi n} e^{-i\pi n} = \frac{i(-1)^n}{2\pi n} \end{aligned}$$

For $n = 0$, we have

$$f_0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} x dx = 0.$$

Thus we have:

$$f(x) = \sum_{n\in\mathbb{Z},n\neq 0} \frac{i(-1)^n}{2\pi n} e^{2\pi inx}$$

For the second equality, we rewrite the sum as:

$$\begin{aligned}
\sum_{n\in\mathbb{Z},n\neq 0} \frac{i(-1)^n}{2\pi n} e^{2\pi inx} &= \sum_{n=1}^{\infty} \frac{i(-1)^n}{2\pi n} e^{2\pi inx} + \sum_{n=-\infty}^{1} \frac{i(-1)^n}{2\pi n} e^{2\pi inx} \\
&= \sum_{n=1}^{\infty} \frac{i(-1)^n}{2\pi n} e^{2\pi inx} + \sum_{n=1}^{\infty} \frac{i(-1)^{-n}}{2\pi(-n)} e^{2\pi i(-n)x} \\
&= \sum_{n=1}^{\infty} \frac{i(-1)^n}{2\pi n} \left[e^{2\pi inx} - e^{-2\pi inx}\right] = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi nx)
\end{aligned}$$

Exercise 5.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} -1 & -\frac{1}{2} < x \leq 0 \\ 1 & 0 < x \leq \frac{1}{2} \end{cases}, \qquad f(x+1) = f(x).$$

a) Show that:

$$f(x) = \frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{2}{2n+1} e^{2\pi i(2n+1)x} = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[2\pi(2n+1)x\right]$$

With convergence in $L^2_{loc.}(\mathbb{R}^n)$.

Solution By Theorem 3.17 and Corollary 3.18 we know that for $f \in L^2_{loc.}(\mathbb{R})$ a periodic function, we have that:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi inx}$$

where the sum converges in $L^2_{loc.}$ and the Fourier coefficients are given by:

$$f_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi inx} f(x)dx.$$

We calculate for $n \neq 0$:

$$\begin{aligned}
f_n &= -\int_{-\frac{1}{2}}^{0} e^{-2\pi inx}dx + \int_{0}^{\frac{1}{2}} e^{-2\pi inx}dx \\
&= \left[\frac{e^{-2\pi inx}}{2\pi ni}\right]_{-\frac{1}{2}}^{0} - \left[\frac{e^{-2\pi inx}}{2\pi ni}\right]_{0}^{\frac{1}{2}} \\
&= \frac{1}{\pi ni}(1-(-1)^n) \\
&= \begin{cases} \frac{2}{\pi ni} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}
\end{aligned}$$

For $n = 0$, we have:

$$f_0 = -\int_{-\frac{1}{2}}^{0} dx + \int_{0}^{\frac{1}{2}} dx = 0.$$

Thus:

$$f(x) = \sum_{n \text{ odd}} \frac{2}{\pi n i} e^{2\pi i n x} = \frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{2}{2n+1} e^{2\pi i(2n+1)x}$$

We can re-write this sum as:

$$\begin{aligned}
\frac{1}{\pi i} \sum_{n=-\infty}^{\infty} \frac{2}{2n+1} e^{2\pi i(2n+1)x} &= \frac{1}{\pi i} \sum_{n=-\infty}^{-1} \frac{2}{2n+1} e^{2\pi i(2n+1)x} + \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{2}{2n+1} e^{2\pi i(2n+1)x} \\
&= \frac{1}{\pi i} \sum_{n=0}^{\infty} -\frac{2}{2n+1} e^{-2\pi i(2n+1)x} + \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{2}{2n+1} e^{2\pi i(2n+1)x} \\
&= \frac{1}{\pi i} \sum_{n=0}^{\infty} \frac{2}{2n+1} \left[e^{2\pi i(2n+1)x} - e^{-2\pi i(2n+1)x} \right] \\
&= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin\left[2\pi(2n+1)x\right]
\end{aligned}$$

Define the partial sum:

$$S_N(x) = 8 \sum_{n=0}^{N-1} \frac{1}{2\pi(2n+1)} \sin\left[2\pi(2n+1)x\right].$$

b) Show that:

$$S_N(x) = 8 \int_0^x \sum_{n=0}^{N-1} \cos\left[2\pi(2n+1)t\right] dt.$$

Solution Noting that:

$$\frac{1}{2\pi(2n+1)} \sin\left[2\pi(2n+1)x\right] = \int_0^x \cos\left[2\pi(2n+1)t\right] dt,$$

we have:

$$\begin{aligned}
S_N(x) &= 8 \sum_{n=0}^{N-1} \int_0^x \cos\left[2\pi(2n+1)t\right] dt \\
&= 8 \int_0^x \sum_{n=0}^{N-1} \cos\left[2\pi(2n+1)t\right] dt.
\end{aligned}$$

where we can commute the sum and the integral since the sum is finite.

c) Show that:

$$\cos\left[2\pi(2n+1)t\right]\sin 2\pi t = \frac{1}{2}\left(\sin\left[2\pi(2n+2)t\right] - \sin\left[4\pi n t\right]\right)$$

And deduce:

$$S_N(x) = 8\int_0^x \frac{\sin 4\pi N t}{2\sin 2\pi t}dt.$$

Solution We have the trig identity:

$$\cos a \sin b = \frac{1}{2}\left[\sin(a+b) - \sin(a-b)\right].$$

Setting $a = 2\pi(2n+1)t$, $b = 2\pi t$ gives the result. Summing from $n = 0$ to $N-1$, we have a telescoping sum, and we find:

$$\sum_{n=0}^{N-1}\cos\left[2\pi(2n+1)t\right]\sin 2\pi t = \frac{1}{2}\sin\left[4\pi N t\right]$$

Dividing by $\sin 2\pi t$ and integrating in t from 0 to x we find:

$$S_N(x) = 8\int_0^x \sum_{n=0}^{N-1}\cos\left[2\pi(2n+1)t\right]dt = 8\int_0^x \frac{\sin 4\pi N t}{2\sin 2\pi t}dt.$$

d) Show that the first local maximum of S_N occurs at $x = \frac{1}{4N}$, and:

$$S_N\left(\frac{1}{4N}\right) \geq 8\int_0^{\frac{1}{4N}} \frac{\sin 4\pi N t}{4\pi t}dt = \frac{2}{\pi}\int_0^{\pi}\frac{\sin s}{s}ds \simeq 1.179\ldots$$

Solution We have that:

$$S_N'(x) = 8\frac{\sin 4\pi N x}{2\sin 2\pi x},$$

which vanishes for $4Nx \in \mathbb{Z}$, with $x \neq 0$. Thus the first local extremum is at $x = \frac{1}{4N}$. This is a maximum, since $S_N'(x) \geq 0$ for $x \leq \frac{1}{4N}$ and $S_N'(x) \leq 0$ for $\frac{1}{4N} \leq x \leq \frac{1}{2N}$. Noticing that for $x \geq 0$ we have $\sin x \leq x$, we find:

$$S_N\left(\frac{1}{4N}\right) = 8\int_0^{\frac{1}{4N}} \frac{\sin 4\pi N t}{2\sin 2\pi t}dt \geq 8\int_0^{\frac{1}{4N}} \frac{\sin 4\pi N t}{4\pi t}dt$$

Changing variables to $s = 4\pi N t$, the second part of the result follows.

e) Conclude that the sum in part a) does not converge uniformly.

Solution For $N \geq 1$, we have that $0 < \frac{1}{4N} < \frac{1}{2}$, so $f\left(\frac{1}{4N}\right) = 1$. Thus for all N we have:

$$\left|S_N\left(\frac{1}{4N}\right) - f\left(\frac{1}{4N}\right)\right| \geq \frac{2}{\pi}\int_0^\pi \frac{\sin s}{s} ds - 1 > 0.$$

Thus:

$$\sup_{x\in(0,\frac{1}{2})} |S_N(x) - f(x)| \not\to 0$$

as $N \to \infty$.

This lack of uniform convergence of a Fourier series at a point of discontinuity is known as Gibbs Phenomenon.

Exercise 5.7. (*) Suppose that $\lambda = \{\lambda_1, \dots \lambda_n\}$ is a basis for $\mathbb{R}^n$. We define the lattice generated by λ to be:

$$\Lambda = \left\{\sum_{j=1}^n z_j \lambda_j : z_j \in \mathbb{Z}\right\}.$$

Define the fundamental cell:

$$q_\Lambda = \left\{\sum_{j=1}^n x_j \lambda_j : |x_j| < \frac{1}{2}\right\}.$$

We say that $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic if:

$$\tau_g u = u \qquad \text{for all } g \in \Lambda.$$

a) Show that there exists $\psi \in C_0^\infty(2q_\Lambda)$ such that $\psi \geq 0$ and

$$\sum_{g\in\Lambda} \tau_g \psi = 1.$$

b) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic and ψ, ψ' are both as in part a), then

$$\frac{1}{|q_\Lambda|}u[\psi] = \frac{1}{|q_\Lambda|}u[\psi'] =: M(u)$$

c) Define the *dual lattice* by:

$$\Lambda^* := \{x \in \mathbb{R}^n : g \cdot x \in 2\pi\mathbb{Z}, \ \forall g \in \Lambda\}$$

Show that there exists a basis $\lambda^* = \{\lambda_1^*, \dots \lambda_n^*\}$ such that $\lambda_j^* \cdot \lambda_k = \delta_{jk}$, and Λ^* is the lattice induced by λ^*.

d) Show that if $g \in \Lambda^*$ then e_g is $\Lambda-$periodic.

e) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$\hat{u} = \sum_{g \in \Lambda^*} c_g \delta_g$$

for some $c_g \in \mathbb{C}$ satisfying $|c_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$.

f) Show that if $u \in \mathscr{D}'(\mathbb{R}^n)$ is $\Lambda-$periodic, then:

$$u = \sum_{g \in \Lambda^*} d_g T_{e_g}$$

where $|d_g| \leq K(1 + |g|)^N$ for some $K > 0$, $N \in \mathbb{Z}$ are given by:

$$d_g = M(e_{-g}u)$$

Exercise 5.8. Suppose $s \geq 0$.

a) Show that $\mathscr{S} \subset H^s(\mathbb{R}^n)$.

Solution Suppose $\phi \in \mathscr{S}$, then $\hat{\phi} \in \mathscr{S}$ as the Fourier transform maps the Schwartz space to itself. Since $\hat{\phi} \in \mathscr{S}$, we know that for any $M \geq 0$ we have:

$$\sup_{x \in \mathbb{R}^n} (1 + |\xi|)^M \left|\hat{\phi}(\xi)\right| < \infty$$

Taking $M = s + \frac{n+1}{2}$, we have that:

$$\left|\hat{\phi}(\xi)\right|^2 \leq \frac{C}{(1 + |\xi|)^{2s+n+1}}$$

Now, we can estimate:

$$\begin{aligned} ||\phi||^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} \left|\hat{f}(\xi)\right|^2 d\xi \\ &\leq \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} \frac{C}{(1 + |\xi|)^{2s+n+1}} d\xi = C \int_{\mathbb{R}^n} \frac{1}{(1 + |\xi|)^{n+1}} d\xi < \infty. \end{aligned}$$

b) Suppose $f \in H^s(\mathbb{R}^n)$. Show that given $\epsilon > 0$ there exists $f_\epsilon \in \mathscr{S}$ with:

$$||f - f_\epsilon||_{H^s(\mathbb{R}^n)} < \epsilon.$$

Hint: First find $g_\epsilon \in \mathscr{S}$ such that

$$\left|\left|(\hat{f} - g_\epsilon)(1 + |\xi|)^s\right|\right|_{L^2(\mathbb{R}^n)} < \epsilon.$$

Solution Since $f \in H^s(\mathbb{R}^n)$, we have that $\hat{f}(1+|\xi|)^s \in L^2(\mathbb{R}^n)$. Thus, given $\epsilon > 0$, there exists $R > 0$ such that:

$$\left|\left|\hat{f}(1+|\xi|)^s\right|\right|_{L^2(\mathbb{R}^n\setminus B_R(0))} < \frac{\epsilon}{2}$$

By Theorem 1.13 b), we can find $h_\epsilon \in C_0^\infty(\mathbb{R}^n)$ such that:

$$\left|\left|\hat{f}(1+|\xi|)^s \mathbb{1}_{B_R(0)} - h_\epsilon\right|\right|_{L^2(\mathbb{R}^n)} < \frac{\epsilon}{2}$$

Defining $g_\epsilon = h_\epsilon(1+|\xi|)^{-s} \in C_0^\infty(\mathbb{R}^n)$, we have:

$$\begin{aligned}\left|\left|(\hat{f}-g_\epsilon)(1+|\xi|)^s\right|\right|_{L^2(\mathbb{R}^n)} &= \left|\left|\hat{f}(1+|\xi|)^s(1-\mathbb{1}_{B_R(0)}) + \hat{f}(1+|\xi|)^s\mathbb{1}_{B_R(0)} - h_\epsilon\right|\right|_{L^2(\mathbb{R}^n)} \\ &\leq \left|\left|\hat{f}(1+|\xi|)^s\right|\right|_{L^2(\mathbb{R}^n\setminus B_R(0))} + \left|\left|\hat{f}(1+|\xi|)^s\mathbb{1}_{B_R(0)} - h_\delta\right|\right|_{L^2(\mathbb{R}^n)} \\ &< \epsilon.\end{aligned}$$

Now, clearly $g_\epsilon \in C_0^\infty(\mathbb{R}^n) \subset \mathscr{S}$, so there exists $f_\epsilon \in \mathscr{S}$ with $\hat{f}_\epsilon = g_\epsilon$. This satisfies:

$$||f - f_\epsilon||_{H^s(\mathbb{R}^n)} = \left|\left|(\hat{f}-g_\epsilon)(1+|\xi|)^s\right|\right|_{L^2(\mathbb{R}^n)} < \epsilon,$$

and we're done.

c) Show that

$$||f||_{H^s(\mathbb{R}^n)} \leq ||f||_{H^t(\mathbb{R}^n)}$$

for $t \geq s$. Deduce that:

$$||f||_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n}\,||f||_{H^s(\mathbb{R}^n)}$$

Hint: Use Parseval's formula

Solution Clearly if $s \leq t$, then:

$$(1+|\xi|)^{2s} \leq (1+|\xi|)^{2t},$$

so we have:

$$\int_{\mathbb{R}^n}(1+|\xi|)^{2s}\left|\hat{f}(\xi)\right|^2 d\xi \leq \int_{\mathbb{R}^n}(1+|\xi|)^{2t}\left|\hat{f}(\xi)\right|^2 d\xi$$

whence it is immediate that:

$$||f||_{H^s(\mathbb{R}^n)} \leq ||f||_{H^t(\mathbb{R}^n)}.$$

Setting $s = 0$, we have that for $t \geq 0$:

$$||f||^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\hat{f}(\xi)\right|^2 d\xi \leq \int_{\mathbb{R}^n}(1+|\xi|)^{2t}\left|\hat{f}(\xi)\right|^2 d\xi$$

which gives the result.

d) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$, where $k = |\alpha|$.

Solution First note that there exists a constant c_α such that:

$$|\xi^\alpha| \leq c_\alpha (1 + |\xi|)^{|\alpha|}$$

Making use of the fact that:

$$\widehat{D^\alpha f}(\xi) = i^{|\alpha|} \xi^\alpha \hat{f}(\xi),$$

we have

$$\begin{aligned} ||D^\alpha f||^2_{H^s(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} \left| \widehat{D^\alpha f}(\xi) \right|^2 d\xi \\ &= \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} |\xi^\alpha|^2 \left| \widehat{f}(\xi) \right|^2 d\xi \\ &\leq c_\alpha \int_{\mathbb{R}^n} (1 + |\xi|)^{2s+2k} \left| \widehat{f}(\xi) \right|^2 d\xi = ||f||^2_{H^{s+k}(\mathbb{R}^n)} . \end{aligned}$$

so that:

$$||D^\alpha f||_{H^s(\mathbb{R}^n)} \leq c_\alpha \, ||f||_{H^{s+k}(\mathbb{R}^n)}$$

and hence the operator $D^\alpha : H^{s+k}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is bounded.

Exercise 5.9. Suppose that $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and that $u(t, x)$ is the solution of the heat equation with initial data u_0. Explicitly, u is given by:

$$u(t, x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}_0(\xi) e^{-t|\xi|^2} e^{i\xi\cdot x} d\xi,$$

for $t > 0$.

a) Show that:

$$||u(t, \cdot)||_{L^2(\mathbb{R}^n)} \leq ||u_0||_{L^2(\mathbb{R}^n)} ,$$

Solution By Parseval's formula, we have that:

$$||u(t, \cdot)||_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} ||\hat{u}(t, \cdot)||_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \left|\left| \hat{u}_0 e^{-t|\xi|^2} \right|\right|_{L^2(\mathbb{R}^n)}$$

However, for $t \geq 0$, we have $\left| e^{-t|\xi|^2} \right| \leq 1$, so we can estimate:

$$\frac{1}{(2\pi)^n} \left|\left| \hat{u}_0 e^{-t|\xi|^2} \right|\right|_{L^2(\mathbb{R}^n)} \leq \frac{1}{(2\pi)^n} ||\hat{u}_0||_{L^2(\mathbb{R}^n)} = ||u_0||_{L^2(\mathbb{R}^n)}$$

where in the last line we applied Parseval again. This gives the result.

b) Show that:

$$u(t,x) = u_0 \star K_t(x)$$

where the *heat kernel* is given by:

$$K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}.$$

Solution First, we recall from Example 12 iv) that if $f(x) = e^{-\frac{1}{2}|x|^2}$, then $\hat{f}(\xi) = (2\pi)^{\frac{n}{2}} e^{-\frac{1}{2}|\xi|^2}$. We also know from Lemma 3.2 i) that if $f_\lambda(x) = \lambda^{-n} f(\lambda^{-1}x)$, then $\hat{f}_\lambda(\xi) = \hat{f}(\lambda\xi)$. Note that:

$$K_t(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \left(\frac{1}{\sqrt{2t}}\right)^n e^{-\frac{1}{2}\left|\frac{x}{\sqrt{2t}}\right|^2}$$

Combining these facts, we deduce that:

$$\widehat{K_t}(\xi) = e^{-t|\xi|^2}.$$

Thus, we have that:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)\widehat{K_t}(\xi).$$

By Lemma 3.2 ii), we have that:

$$u(t,x) = u_0 \star K_t(x) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u_0(y) e^{-\frac{|x-y|^2}{4t}} dy$$

c) Suppose that $u_0 \geq 0$. Show that $u \geq 0$, and:

$$||u(t,\cdot)||_{L^1(\mathbb{R}^n)} = ||u_0||_{L^1(\mathbb{R}^n)}.$$

[Hint: Lemma 1.9 may be useful]

Solution By Lemma 1.9, we know that if $f, g \in \mathscr{S}$:

$$\int_{\mathbb{R}^n} f \star g(x) dx = \int_{\mathbb{R}^n} f(x) dx \int_{\mathbb{R}^n} g(x) dx.$$

Suppose $u_0 \in \mathscr{S}$. Since $u_0 \geq 0$, and $K_t \geq 0$ we have $u_0 \star K_t \geq 0$, so that:

$$||u(t,\cdot)||_{L^1(\mathbb{R}^n)} = ||u_0 \star K_t||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u_0 \star K_t(x) dx.$$

Now, applying the result above we have that:

$$||u(t,\cdot)||_{L^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} u_0(x) dx \int_{\mathbb{R}^n} K_t(x) dx = ||u_0||_{L^1(\mathbb{R}^n)}$$

where we have used that:

$$\int_{\mathbb{R}^n} K_t(x) dx = \widehat{K_t}(0) = 1.$$

Now, since $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, we can extend this result to the full space by continuity.

Exercise 5.10. Consider the Schrödinger equation:

$$\begin{cases} u_t = i\Delta u & \text{in } (0,T)\times\mathbb{R}^n, \\ u = u_0 & \text{on } \{0\}\times\mathbb{R}^n \end{cases} \tag{4}$$

Suppose $u_0 \in H^2(\mathbb{R}^n)$.

a) Show that (3) admits a unique solution u such that

$$\boldsymbol{u} \in C^0([0,T); H^2(\mathbb{R}^n)) \cap C^1((0,T); L^2(\mathbb{R}^n)),$$

whose spatial Fourier-Plancherel transform is given by:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)e^{-it|\xi|^2}.$$

Solution Assuming we have such a solution, and Fourier transforming in the spatial variable (which we're entitled to do, under the assumptions on u) the equation becomes:

$$\begin{cases} \hat{u}_t = -i\,|\xi|^2\,\hat{u} & \text{in } (0,T)\times\mathbb{R}^n, \\ \hat{u} = \hat{u}_0 & \text{on } \{0\}\times\mathbb{R}^n \end{cases}$$

This can be readily solved to give:

$$\hat{u}(t,\xi) = \hat{u}_0(\xi)e^{-it|\xi|^2}. \tag{5}$$

Now suppose $u_0 \in H^2(\mathbb{R}^n)$. The u defined by (**??**) belongs to the space $C^0([0,T); H^2(\mathbb{R}^n))\cap C^1((0,T); L^2(\mathbb{R}^n))$ and we can undo the steps above to deduce that u solves (3).

Bonus material: We'll verify that $u \in C^0([0,T); H^2(\mathbb{R}^n))\cap C^1((0,T); L^2(\mathbb{R}^n))$. This is included for interest, but would be overkill in an exam. First, we note that:

$$||u(t,\cdot)||^2_{H^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\hat{u}(t,\xi)|^2 (1+|\xi|)^4 d\xi = \int_{\mathbb{R}^n} |\hat{u}_0(\xi)|^2 (1+|\xi|)^4 d\xi = ||u_0||^2_{H^2(\mathbb{R}^n)}$$

so that $u(t,\cdot) \in H^2(\mathbb{R}^n)$. We can verify that in fact the map $\boldsymbol{u} : [0,T) \to H^2(\mathbb{R}^n)$ is continuous. We calculate, for $t, t' \in [0,T)$:

$$\left|\left|u(t,\cdot) - u(t',\cdot)\right|\right|^2_{H^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|e^{-it|\xi|^2} - e^{-it'|\xi|^2}\right|^2 |\hat{u}_0(\xi)|^2 (1+|\xi|)^4 d\xi$$

Clearly:

$$\lim_{t'\to t} \left|e^{-it|\xi|^2} - e^{-it'|\xi|^2}\right|^2 |\hat{u}_0(\xi)|^2 (1+|\xi|)^4 = 0$$

and moreover

$$\left|e^{-it|\xi|^2} - e^{-it'|\xi|^2}\right|^2 |\hat{u}_0(\xi)|^2 (1+|\xi|)^4 \leq 2\,|\hat{u}_0(\xi)|^2 (1+|\xi|)^4$$

which is integrable, since $u_0 \in H^2(\mathbb{R}^n)$. Thus we conclude that:

$$\lim_{t'\to t} \left|\left|u(t,\cdot) - u(t',\cdot)\right|\right|_{H^2(\mathbb{R}^n)} = 0.$$

Next we verify that the map $\boldsymbol{u} : (0,T) \to L^2(\mathbb{R}^n)$ is differentiable. Let $w(t,x)$ be defined in terms of its spatial Fourier transform by:

$$\hat{w}(t,\xi) = -i\,|\xi|^2\,\hat{u}_0(\xi)e^{-it|\xi|^2}.$$

First we claim that $w(t,\cdot) \in L^2(\mathbb{R}^n)$. By Parseval's formula, we have:

$$||w(t,\cdot)||^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\,||\hat{w}(t,\cdot)||^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\hat{u}_0(\xi)|^2\,|\xi|^4\,d\xi$$

which is finite since $u_0 \in H^2(\mathbb{R}^n)$. Next we claim $\boldsymbol{w} : (0,T) \to L^2(\mathbb{R}^n)$ given by $t \mapsto w(t,\cdot)$ is in fact continuous. To see this we calculate, again with Parseval:

$$\left|\left|w(t,\cdot) - w(t',\cdot)\right|\right|^2_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|e^{-it|\xi|^2} - e^{-it'|\xi|^2}\right|^2 |\hat{u}_0(\xi)|^2\,|\xi|^4\,d\xi$$

Invoking the Dominated convergence theorem precisely as above shows that:

$$\lim_{t'\to t}\left|\left|w(t,\cdot) - w(t',\cdot)\right|\right|_{L^2(\mathbb{R}^n)} \to 0.$$

Finally, for $t, t' \in (0,T)$ we wish to show that:

$$\lim_{t'\to t}\frac{u(t,\cdot) - u(t',\cdot)}{t-t'} \to w(t,\cdot).$$

as $t' \to t$. We calculate:

$$\begin{aligned}\frac{\hat{u}(t,\cdot) - \hat{u}(t',\cdot)}{t-t'} + i\,|\xi|^2\,\hat{u}_0(\xi)e^{-it|\xi|^2} &= \hat{u}_0(\xi)\left(\frac{e^{-it|\xi|^2} - e^{-it'|\xi|^2}}{t-t'} + i\,|\xi|^2\,e^{-it|\xi|^2}\right)\\ &= \hat{u}_0(\xi)\,|\xi|^2\left(\frac{e^{-it|\xi|^2} - e^{-it'|\xi|^2}}{t\,|\xi|^2 - t'\,|\xi|^2} + ie^{-it|\xi|^2}\right)\end{aligned}$$

Note that:

$$e^{-it|\xi|^2} - e^{-it'|\xi|^2} = -i\int_{t'|\xi|^2}^{t|\xi|^2} e^{-is}ds$$

so that

$$\left|e^{-it|\xi|^2} - e^{-it'|\xi|^2}\right| \leq \left|t\,|\xi|^2 - t'\,|\xi|^2\right|.$$

We conclude that:

$$\left|\left(\frac{e^{-it|\xi|^2} - e^{-it'|\xi|^2}}{t\,|\xi|^2 - t'\,|\xi|^2} + ie^{-it|\xi|^2}\right)\right| \leq 2$$

and moreover

$$\left(\frac{e^{-it|\xi|^2} - e^{-it'|\xi|^2}}{t\,|\xi|^2 - t'\,|\xi|^2} + ie^{-it|\xi|^2}\right) \to 0$$

as $t' \to t$.

In conclusion, we can deduce that:

$$\left| \frac{\hat{u}(t,\cdot) - \hat{u}(t',\cdot)}{t - t'} + i\left|\xi\right|^2 \hat{u}_0(\xi)e^{-it|\xi|^2} \right|^2 \to 0$$

as $t' \to t$ and moreover:

$$\left| \frac{\hat{u}(t,\cdot) - \hat{u}(t',\cdot)}{t - t'} + i\left|\xi\right|^2 \hat{u}_0(\xi)e^{-it|\xi|^2} \right|^2 \leq \left|\hat{u}_0\right|^2 \left|\xi\right|^4$$

where the right hand side is integrable since $u_0 \in H^2(\mathbb{R}^n)$. Thus we can invoke the Dominated Convergence Theorem to decuce:

$$\left|\left| \frac{u(t,\cdot) - u(t',\cdot)}{t - t'} - w(t,\cdot) \right|\right|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^n} \left|\left| \frac{\hat{u}(t,\cdot) - \hat{u}(t',\cdot)}{t - t'} - \hat{w}(t,\cdot) \right|\right|_{L^2(\mathbb{R}^n)} \to 0$$

as $t' \to t$.

b) Show that:

$$||u(t,\cdot)||_{H^2(\mathbb{R}^n)} = ||u_0||_{H^2(\mathbb{R}^n)}$$

Solution This is a simple calculation with the norm:

$$||u(t,\cdot)||^2_{H^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \left|\hat{u}(t,\xi)\right|^2 (1 + |\xi|)^4 d\xi = \int_{\mathbb{R}^n} \left|\hat{u}_0(\xi)\right|^2 (1 + |\xi|)^4 d\xi = ||u_0||^2_{H^2(\mathbb{R}^n)}$$

*c) For $t > 0$, let $K_t \in L^1_{loc.}(\mathbb{R}^n)$ be given by:

$$K_t(x) = \frac{1}{(4\pi i t)^{\frac{n}{2}}} e^{\frac{i|x|^2}{4t}},$$

where for n odd we take the usual branch cut so that $i^{\frac{1}{2}} = e^{i\frac{\pi}{4}}$. For $\epsilon > 0$ set $K_t^\epsilon(x) = e^{-\epsilon|x|^2} K_t(x)$.

i) Show that $T_{K_t^\epsilon} \to T_{K_t}$ in $\mathscr{S}'$ as $\epsilon \to 0$.

ii) Show that if $\Re(\sigma) > 0$, then:

$$\int_{\mathbb{R}} e^{-\sigma x^2 - ix\xi} dx = \sqrt{\frac{\pi}{\sigma}} e^{-\frac{\xi^2}{4\sigma}}.$$

iii) Deduce that

$$\widehat{K_t^\epsilon}(\xi) = \left(\frac{1}{1 + 4it\epsilon}\right)^{\frac{n}{2}} e^{\frac{-it|\xi|^2}{1+4it\epsilon}}$$

iv) Conclude that:

$$\widehat{T_{K_t}} = T_{\tilde{K}_t},$$

where $\tilde{K}_t = e^{-it|\xi|^2}$.

*d) Suppose that $u \in \mathscr{S}(\mathbb{R}^n)$. Show that for $t > 0$:

$$u(t, x) = \int_{\mathbb{R}^n} u_0(y) K_t(x - y) dy,$$

and deduce:

$$\sup_{t>0, x\in\mathbb{R}^n} |u(t, x)| \leq \frac{1}{(4\pi t)^{\frac{n}{2}}} ||\hat{u}_0||_{L^1(\mathbb{R}^n)} .$$

This type of estimate which shows us that (locally) solutions to the Schrödinger equation decay in time is known as a *dispersive estimate.*

Exercise 5.11. Let $\mathbb{R}^3_* := \mathbb{R}^3 \setminus \{0\}$, $S_{*,T} := (-T, T) \times \mathbb{R}^3_*$ and $|x| = r$. You may assume the result that if $u = u(r, t)$ is radial, we have

$$\Delta u(|x| , t) = \Delta u(r, t) = \frac{\partial^2 u}{\partial r^2}(r, t) + \frac{2}{r}\frac{\partial u}{\partial r}(r, t)$$

a) Suppose $u(x, t) = \frac{1}{r} v(r, t)$ for some function v. Show that u solves the wave equation on $\mathbb{R}^3_* \times (0, T)$ if and only if v satisfies the one-dimensional wave equation

$$-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2} = 0$$

on $(0, \infty) \times (-T, T)$.

Solution Assuming u has the form given, we calculate

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2} v$$

and

$$\frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2} - 2\frac{1}{r^2}\frac{\partial v}{\partial r} + \frac{2}{r^3} v$$

so that

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial^2 v}{\partial r^2}.$$

We also have that

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{r}\frac{\partial^2 v}{\partial t^2}$$

so that

$$-\frac{\partial^2 u}{\partial t^2} + \Delta u = \frac{1}{r}\left(-\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2 v}{\partial r^2}\right)$$

so clearly, u obeys the three dimensional wave equation away from $r = 0$ if and only if v obeys the one-dimensional wave equation.

b) Suppose $f, g \in C_c^2(\mathbb{R})$. Deduce that

$$u(x,t) = \frac{f(r+t)}{r} + \frac{g(r-t)}{r}$$

is a solution of the wave equation on $S_{*,T}$ which vanishes for large $|x|$.

Solution This function is of the form previously considered, with

$$v(r,t) = f(r+t) + g(r-t).$$

This solves the one-dimensional wave equation as can be verified directly. Moreover, since f, g have compact support, for any fixed t we find that u will vanish for sufficiently large r.

c) Show that if $f \in C_c^3(\mathbb{R})$ is an odd function (i.e. $f(s) = -f(-s)$ for all s) then

$$u(x,t) = \frac{f(r+t) + f(r-t)}{2r}$$

extends as a C^2 function which solves the wave equation on $S_T := (-T,T) \times \mathbb{R}^3$, with

$$u(0,t) = f'(t).$$

Solution Using the fact that f is three times differentiable at t, we have

$$f(r+t) = f(t) + f'(t)r + \frac{1}{2}f''(t)r^2 + \frac{1}{6}f'''(t)r^3 + R(r,t)$$

where

$$\lim_{r \to 0} \frac{R(t,r)}{r^3} = 0.$$

Making use of a similar expansion about $-t$ and Inserting the condition that f is odd, we deduce

$$\begin{aligned} u(x,t) &= \frac{f'(t)r + \frac{1}{6}f'''(t)r^3 + \hat{R}(r,t)}{r} = f'(t) + \frac{1}{6}f'''(t)r^2 + \tilde{R}(r,t) \\ &= f'(t) + \frac{1}{6}f'''(t)\delta_{ij}x^i x^j + \tilde{R}(|x|,t) \end{aligned}$$

with $\tilde{R}(|x|,t) \in C^2(S_{*,T})$ and

$$\lim_{x \to 0} \frac{\tilde{R}(|x|,t)}{|x|^2} = 0.$$

This implies that $\tilde{R}(\cdot,t) \in C^2(\mathbb{R}^3)$ and hence $u(\cdot,t) \in C^2(\mathbb{R}^3)$. A similar calculation establishes that $u_t(\cdot,t) \in C^1(\mathbb{R}^3)$ and $u_{tt}(\cdot,t) \in C^0(\mathbb{R}^3)$, which completes the proof.

*d) By considering a suitable sequence of functions f, or otherwise, deduce that there exists no constant C independent of u such that the estimate

$$\sup_{S_T}(|u| + |u_t|) \leq C \sup_{\Sigma_0}(|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$.

Solution Suppose that $\chi : \mathbb{R} \to \mathbb{R}$ satisfies:

a) χ is smooth and even.

b) $\chi(0) = 1$

c) $\chi(s) = 0$ for $|s| > 1$.

By the mean value theorem, we can easily see that $\sup_{\mathbb{R}} |\chi'| \geq 1$. Consider the sequence of functions:

$$f_k(s) = \frac{1}{\sqrt{k}} \left[\chi\left(ks - 2\sqrt{k}\right) - \chi\left(ks + 2\sqrt{k}\right)\right]$$

for k a sufficiently large integer. We see that $f_k(s)$ is odd, smooth, and satisfies

$$\sup_{0<s<2k^{-\frac{1}{2}}} \left|f_k'(s)\right| \geq \sqrt{k}, \qquad \sup_{0<s} \left|\frac{f_k(s)}{s}\right| \leq C.$$

Now let us construct a solution to the wave equation from f_k as in part c). Clearly, we have

$$\sup_{\Sigma_0} |u| = \sup_{s \geq 0} \left|\frac{f_k(s)}{s}\right| \leq C,$$

and

$$\sup_{S_T} |u| \geq \sup_{-T<t<T} |u(0,t)| \geq \sup_{0<s<2k^{-\frac{1}{2}}} \left|f_k'(s)\right| \geq \sqrt{k}.$$

Where we take k sufficiently large that $2k^{-\frac{1}{2}} < T$. We also note that $u_t|_{\Sigma_0} = 0$. Suppose now that there exists a constant C such that

$$\sup_{S_T} |u| \leq C \sup_{\Sigma_0}(|u| + |u_t|)$$

holds for all solutions $u \in C^2(S_T)$ of the wave equation which vanish for large $|x|$. Applying this to the sequence we have just constructed we find

$$\sqrt{k} \leq C$$

for all sufficiently large k, which is clearly absurd.

Exercise A.1. Suppose that $\lambda_1\lambda_2 \geq 0$ and that $U \subset X$ is a convex subset of a vector space X. Show that:

$$\lambda_1 U + \lambda_2 U = (\lambda_1 + \lambda_2)U.$$

Exercise A.2. a) Suppose that (S, τ) is a topological space, and that β is a base for τ. Show that:

i) If $x \in S$, then there exists some $B \in \beta$ with $x \in B$.

ii) If B_1, $B_2 \in \beta$, then for every $x \in B_1 \cap B_2$ there exists $B \in \beta$ with:

$$x \in B \qquad B \subset B_1 \cap B_2.$$

b) Conversely, suppose that one is given a set S and a collection β of subsets of S satisfying $i)$, $ii)$ above. Define τ by:

$$U \in \tau \iff \text{for all } x \in U, \text{ there exists } B \in \beta \text{ such that } x \in B \text{ and } B \subset U.$$

i.e. τ is the set of all unions of elements of β. Show that (S, τ) is a topological space, with base β. We say that τ is the topology generated by β

c) Suppose that β, β' both satisfy conditions $i)$, $ii)$ above and generate topologies τ, τ' respectively. Moreover, suppose that if $B \in \beta$ then for every $x \in B$ there exists $B' \in \beta'$ satisfying

$$x \in B', \qquad \text{and} \qquad B' \subset B$$

Then $\tau \subset \tau'$.

Solution a) i) Since we have $S \in \tau$, and that any element of τ can be written as a union of elements of β, we must have that for each $x \in S$ there is some $B \in \beta$ with $x \in B$.

ii) Since β is a base, if $B_1, B_2 \in \beta$ they are open, and hence so is $B_1 \cap B_2$. Since every open set is a union of elements of β, for any point $x \in B_1 \cap B_2$, there must be some $B \in \beta$ with $x \in B$ and $B \subset B_1 \cap B_2$.

b) The empty set belongs to τ trivially. S belongs to τ as a consequence of condition $i)$. Suppose that $\{U_i\}_{i\in\mathcal{I}}$ is a set of elements of τ indexed by a set $\mathcal{I}$. We need to show:

$$U = \bigcup_{i\in\mathcal{I}} U_i \in \tau.$$

Suppose $x \in U$. Then there is some $i \in \mathcal{I}$ such that $x \in U_i$. Since $U_i \in \tau$, there exists some $B \in \beta$ with $x \in B$ and $B \subset U_i$. But then since $U_i \subset U$, we have $B \subset U$ and so τ is closed under arbitrary unions. It remains to show then that the intersection of

Please send any corrections to c.warnick@imperial.ac.uk two elements of τ belong to τ. Suppose $U_1, U_2 \subset \tau$, and let $x \in U_1 \cap U_2$. Then by definition of τ, there exist $B_1, B_2 \in \beta$ with $x \in B_1$, $x \in B_2$ and $B_1 \subset U_1$, $B_2 \subset U_2$. By condition ii), there exists $B \in \beta$ with $x \in B$ and $B \subset B_1 \cap B_2 \subset U_1 \cap U_2$, so $U_1 \cap U_2 \in \tau$ and τ is closed under pairwise intersection and we're done.

c) Suppose that $U \in \tau$, and let $x \in U$. Then by the definition of τ there exists $B \in \beta$ with $B \subset U$ and $x \in B$. By the assumption, there exists $B' \in \beta'$ with $x \in B'$ and $B' \subset B \subset U$, thus $U \in \tau'$ and we're done.

Exercise A.3. Suppose (S_1, τ_1), (S_2, τ_2) and (S_3, τ_3) are topological spaces, and that $f : S_1 \times S_2 \to S_3$ is a continuous map. Show that for each $a \in S_1$ and $b \in S_2$, the maps

$$f_a \; : \; S_2 \to S_3, \quad y \mapsto f(a, y), \qquad\qquad f^b \; : \; S_1 \to S_3, \quad x \mapsto f(x, b),$$

are continuous.

The condition that f is continuous with respect to the product topology is sometimes called *joint continuity*, while the continuity of f_a, f^b is called *separate continuity*. Thus joint continuity implies separate continuity. The converse is not true.

Solution Suppose that $U \subset S_3$ is open, then by the definition of continuity, we know that $f^{-1}(U) \subset S_1 \times S_2$ is open in the product topology. Thus for each (x, y) such that $f(x, y) \in U$, we can find neighbourhoods U_1, U_2 of x, y respectively such that $U_1 \times U_2 \subset f^{-1}(U)$.

Now fix a and consider $y \in f_a^{-1}(U)$. By the previous argument there exists U_1, U_2 open in S_1, S_2 respectively such that $a \in U_1$, $y \in U_2$ and for any $(x', y') \in U_1 \times U_2$ we have $f(x', y') \in U$. In particular this holds for $x' = a$ and $y' \in U_2$. Thus the set U_2 is open in S_2 and $U_2 \subset f_a^{-1}(U)$. The case of f^b follows by an identical argument.

Exercise A.4. Show that the base

$$\beta_{\mathbb{Q}} = \{(p, q) : p, q \in \mathbb{Q}, \; p < q\},$$

generates the standard topology on $\mathbb{R}$.

Solution

1. Since $\beta_{\mathbb{Q}} \subset \beta_{\mathbb{R}}$, it is clear that any open set with respect to the topology generated by $\beta_{\mathbb{Q}}$ is also an open set with respect to the topology generated by $\beta_{\mathbb{R}}$.

2. To show the converse, we must show that if $B \in \beta_{\mathbb{R}}$ and $x \in B$ is arbitrary, then there exists $B' \in \beta_{\mathbb{Q}}$ with $x \in B'$ and $B' \subset B$. To see this, recall that between any two real numbers lies an element of $\mathbb{Q}$. Thus if $B = (a, b)$ and $x \in (a, b)$, there exists $p, q \in \mathbb{Q}$ with $a < p < x < q < b$. taking $B' = (p, q)$, we're done.

Exercise A.5. Suppose that (S, d) is a metric space. Show that S is Hausdorff with respect to the metric topology.

Solution Suppose $x, y \in S$ with $x \neq y$, and let $r = d(x, y)$. Consider the sets $U_1 = B_{\frac{r}{3}}(x)$, $U_2 = B_{\frac{r}{3}}(y)$. By definition of the metric topology these are both open sets, and moreover $x \in U_1$ and $y \in U_2$. By the triangle inequality we have:

$$r = d(x, y) \leq d(x, z) + d(y, z)$$

Thus if $z \in U_1$, we must have $d(y, z) > \frac{2r}{3}$, so that $z \notin U_2$, thus $U_1 \cap U_2 = \emptyset$ and we have shown that S is Hausdorff with the metric topology.

Exercise A.6. Let us take $X = \mathbb{R}^n$, thought of as a vector space over $\mathbb{R}$ and define:

$$||(x_1, \ldots, x_n)||_p = (|x_1|^p + \ldots |x_n|^p)^{\frac{1}{p}}, \qquad p \geq 1.$$

a) Show that $(\mathbb{R}^n, ||\cdot||_p)$ is a normed vector space:

i) First check that the positivity and homogeneity property are satisfied.

ii) Establish the triangle inequality for the special case $p = 1$.

iii) Next prove Young's inequality: if $a, b \in \mathbb{R}_+$ and $p, q > 1$ with $p^{-1} + q^{-1} = 1$ then:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Hint: set $t = p^{-1}$, *consider the function* $\log\left[ta^p + (1-t)b^q\right]$ *and use the concavity of the logarithm*

iv) With $p, q > 1$ such that $p^{-1} + q^{-1} = 1$, show that if $||x||_p = 1$ and $||y||_q = 1$ then

$$\sum_{i=1}^{n} |x_i y_i| \leq 1.$$

Deduce Hölder's inequality:

$$\sum_{i=1}^{n} |x_i y_i| \ \leq ||x||_p \, ||y||_q, \qquad \text{for all } x, y, \in \mathbb{R}^n.$$

v) Show that

$$||x + y||_p^p \leq \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}$$

vi) Apply Hölder's inequality to deduce:

$$||x + y||_p^p \leq \left(||x||_p + ||y||_p\right) ||x + y||_p^{p-1}$$

and conclude

$$||x + y||_p \leq ||x||_p + ||y||_p.$$

b) Show that the metric topology of $(\mathbb{R}^n, ||\cdot||_p)$ agrees with the standard topology.
Hint: Use part c) of Exercise A.2

Solution a) i) We fist note that $||x||_p \geq 0$, with equality if and only if $x_1 = x_2 = \cdots = x_n = 0$. The homogeneity property $||ax|| = |a|\,||x||$. is easily verified.

ii) For the case $p = 1$, the triangle inequality follows easily using the fact that for $a, b \in \mathbb{R}$ we have $|a+b| \leq |a|+|b|$ as can be shown by considering the four possible choices of sign for a, b separately.

iii) Since $p > 1$, setting $t = p^{-1}$ we have $0 < t < 1$. Now recall that the logarithm function is concave. This implies that if $0 < x, y$, and $0 < t < 1$ then:

$$t \log x + (1-t) \log y \leq \log(tx + (1-t)y).$$

Applying this with $x = a^p$, $y = b^q$ we have for the LHS:

$$t \log x + (1-t) \log y = \frac{1}{p} \log a^p + \frac{1}{q} \log b^q = \log (ab)\,.$$

and $tx + (1-t)y = \frac{a^p}{p} + \frac{b^q}{q}$. Thus:

$$\log (ab) \leq \log \left(\frac{a^p}{p} + \frac{b^q}{q} \right).$$

The result follows from exponentiating and using the monotonicity of the exponential.

iv) We apply Young's inequality to obtain:

$$\sum_{i=1}^{n} |x_i y_i| \leq \sum_{i=1}^{n} \left(\frac{|x_i|^p}{p} + \frac{|y_i|^q}{q} \right).$$

Next we use that $||x||_p = 1$ to deduce that

$$1 = ||x||_p^p = \sum_{i=1}^{n} |x_i|^p$$

and similarly for y, so that:

$$\sum_{i=1}^{n} |x_i y_i| \leq \frac{1}{p} + \frac{1}{q} = 1.$$

Now suppose $x, y \in \mathbb{R}^n$. If either $x = 0$ or $y = 0$ then Hölder's inequality holds trivially, so we can suppose $||x||_p \neq 0$ and $||y||_q \neq 0$. We define:

$$x' = \frac{1}{||x||_p} x, \qquad y' = \frac{1}{||y||_q} y.$$

These satisfy $||x'||_p = ||y'||_q = 1$, so we can apply the first result to obtain:

$$1 \geq \sum_{i=1}^{n} |x_i' y_i'| = \frac{1}{||x||_p \, ||y||_q} \sum_{i=1}^{n} |x_i y_i|$$

Multiplying by $||x||_p \, ||y||_q$ we are done.

v) We estimate

$$\begin{aligned} ||x+y||_p^p &= \sum_{i=1}^{n} |x_i + y_i|^p = \sum_{i=1}^{n} |x_i + y_i| \, |x_i + y_i|^{p-1} \\ &\leq \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} + \sum_{i=1}^{n} |y_i| \, |x_i + y_i|^{p-1}, \end{aligned}$$

using the triangle inequality for $|\cdot|$.

vi) We apply Hölder's inequality to each sum separately. We have that $q^{-1} = 1 - p^{-1}$ so that $q = \frac{p}{p-1}$

$$\begin{aligned} \sum_{i=1}^{n} |x_i| \, |x_i + y_i|^{p-1} &\leq ||x||_p \left[\sum_{i=1}^{n} \left(|x_i + y_i|^{p-1} \right)^q \right]^{\frac{1}{q}} \\ &= ||x||_p \left[\sum_{i=1}^{n} |x_i + y_i|^p \right]^{\frac{p-1}{p}} \\ &= ||x||_p \, ||x - y||_p^{p-1} \end{aligned}$$

b) Denote by τ the standard topology on $\mathbb{R}^n$. Recall that a base is given by the sets $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$ for real numbers a_i, b_i with $a_i < b_i$, while a base for the norm topology is given by the balls $B_r(x) = \{y \in \mathbb{R}^n : ||y - x||_p < r\}$. Suppose

$$B = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n),$$

and that $a_i < x_i < b_i$, so that $x \in B$. Then if y satisfies $|y_i - x_i| < \min\{-a_i, b_i\}$, we have $y \in B$. Thus there exists $\epsilon > 0$ such that if $\max_i |y_i - x_i| < \epsilon$, we have $x \in B$. Now, note that

$$\max_i |y_i - x_i| \leq ||x||_p,$$

so that if $y \in B_\epsilon(x)$, we must have $y \in B$. Thus $B_\epsilon(x) \subset B$. This proves that $\tau \subset \tau_p$. Conversely, suppose that $y \in B_r(x)$ and suppose that $z \in (y_1 - \delta, y_1 + \delta) \times \cdots (y_n - \delta, y_n + \delta)$ for some $\delta > 0$. We then have:

$$||y - z||_p \leq n^{\frac{1}{p}} \delta$$

By the triangle inequality we have

$$||z - x||_p \leq ||z - y||_p + ||y - x||_p$$

Now, since $y \in B_r(x)$, there exists $\eta > 0$ such that $||y - x||_p \leq r - \eta$. Taking $\delta = \frac{\eta}{2n^{\frac{1}{p}}}$ we conclude

$$||z - x||_p \leq r - \frac{\eta}{2}$$

and so $z \in B_r(x)$ and $(y_1 - \delta, y_1 + \delta) \times \cdots (y_n - \delta, y_n + \delta) \subset B_r(x)$. Thus $\tau_p \subset \tau$

Exercise A.7 ($\star$)**.** Let $X = C[0,1]$, the set of continuous functions on the closed interval $[0,1]$. For $f \in X$, $p \geq 0$ define:

$$||f||_p = \left(\int_0^1 |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

a) Show that X is a vector space over $\mathbb{R}$, where scalar multiplication and vector addition are defined pointwise.

b) Establish Hölder's inequality:

$$||fg||_1 \leq ||f||_p \, ||g||_q$$

for $p, q > 1$ with $p^{-1} + q^{-1} = 1$.

c) Show that $(X, ||\cdot||_p)$ is a normed space.

d) Suppose $p \leq p'$. Show that:

$$||f||_p \leq ||f||_{p'}$$

e) Let τ_p be the metric topology of $(X, ||\cdot||_p)$. Show that if $p \leq p'$:

$$\tau_p \subset \tau_{p'}.$$

f) Consider the sequence of functions:

$$f_n(x) = \begin{cases} n^{\gamma - 1} & 0 \leq x < \frac{1}{n} \\ \frac{1}{n} x^{-\gamma} & \frac{1}{n} \leq x \leq 1 \end{cases}$$

where $n = 1, 2, \ldots$

i) Show that $f_n \in C[0,1]$ and

$$\lim_{n \to \infty} ||f_n||_p = \begin{cases} 0 & \gamma < \frac{p+1}{p} \\ \left(\frac{p+1}{p}\right)^{\frac{1}{p}} & \gamma = \frac{p+1}{p} \\ \infty & \gamma > \frac{p+1}{p} \end{cases}$$

ii) By choosing γ carefully, show that if $p < p'$ then

$$\tau_{p'} \not\subset \tau_p.$$

Hint: in parts b), c) follow the same steps as for the finite dimensional case in Exercise A.6.

Solution a) If $f, g \in C[0,1]$ and $\lambda \in \mathbb{R}$, then defining $h : x \mapsto f(x) + \lambda g(x)$ we have $h \in C[0,1]$. The distributive property follows straightforwardly.

b) First suppose that $f, g \in C[0,1]$ with $||f||_p = ||g||_q = 1$. We calculate

$$\begin{aligned} ||fg||_1 &= \int_0^1 f(x)g(x)dx \\ &\leq \int_0^1 \left(\frac{f(x)^p}{p} + \frac{g(x)^q}{q} \right) dx \\ &= \frac{||f||_p^p}{p} + \frac{||g||_q^q}{q} \\ &= \frac{1}{p} + \frac{1}{q} = 1. \end{aligned}$$

The case of general f, g follows by scaling. If f or g is identically zero then the result is trivially true, so can assume that f is non-zero at some point $x_0 \in [0,1]$. By the continuity of f, we must have that $|f(x)| > |f(x_0)|/2$ on some open interval $(x_0 - \epsilon, x_0 + \epsilon)$ with $\epsilon > 0$. Thus we can estimate $||f||_p > \epsilon |f(x_0)| > 0$. Similarly, if g is not identically zero then $||g||_q > 0$. Thus we can define $F(x) = f(x)/\,||f||_p$, $G(x) = g(x)/\,||g||_q$, which have $||F||_p = 1$, $||G||_q = 1$. Applying the result above, we quickly deduce Hölder's inequality.

c) The homogeneity property is straightforward to verify. Next assume that f is non-zero at some point $x_0 \in [0,1]$. By the continuity of f, we must have that $|f(x)| > |f(x_0)|/2$ on some open interval $(x_0 - \epsilon, x_0 + \epsilon)$ with $\epsilon > 0$. Thus we can estimate $||f||_p > \epsilon |f(x_0)| > 0$. Thus $||f||_p = 0$ if and only if $f \equiv 0$. To show the triangle inequality, we estimate

$$\begin{aligned} ||f+g||_p^p &= \int_0^1 |f(x)+g(x)|^p \, dx = \int_0^1 |f(x)+g(x)| \, |f(x)+g(x)|^{p-1} \, dx \\ &\leq \int_0^1 \left(|f(x)| \, |f(x)+g(x)|^{p-1} + |g(x)| \, |f(x)+g(x)|^{p-1} \right) dx \end{aligned}$$

Now, we apply Hölder's inequality to find:

$$\begin{aligned} \int_0^1 |f(x)| \, |f(x)+g(x)|^{p-1} \, dx &\leq ||f||_p \left|\left| |f+g|^{p-1} \right|\right|_{\frac{p}{p-1}} \\ &= ||f||_p \, ||f+g||_p^{p-1} \end{aligned}$$

and similarly with f and g swapped. Thus we have

$$||f+g||_p^p \leq \left(||f||_p + ||g||_p \right) ||f+g||_p^{p-1}$$

whence the triangle inequality for the norm follows.

d) Since $\frac{p'}{p} > 1$, we can apply the Hölder inequality as follows:

$$\begin{aligned} ||f||_p^p &= |||f|^p||_1 \\ &\leq |||f|^p||_{\frac{p'}{p}} \, ||1||_{\frac{p'}{p'-p}} \\ &= ||f||_{p'}^p \, . \end{aligned}$$

Here we have used that $||1||_q = 1$. Taking the p^{th} root of both sides we are done.

e) Suppose that $U \in \tau_p$. Given $g \in U$, we can find $\epsilon > 0$ such that

$$\{g : ||g - f||_p < \epsilon\} \subset U.$$

Now, since $||g - f||_p \leq ||g - f||_{p'}$, we have that

$$\{g : ||g - f||_{p'} < \epsilon\} \subset \{g : ||g - f||_p < \epsilon\}.$$

Thus given $g \in U$, we have found $\epsilon > 0$ such that

$$\{g : ||g - f||_{p'} < \epsilon\} \subset U,$$

which is precisely the condition that $U \in \tau_{p'}$.

f) i) The continuity is straightforward to check, since we just need to verify that there is no jump at $x = n^{-1}$, which indeed there is not. We calculate (for $\gamma \neq p^{-1}$)

$$\begin{aligned} ||f_n||_p^p &= \frac{1}{n}\left(n^{\gamma-1}\right)^p + \int_{\frac{1}{n}}^1 \left(\frac{1}{n}x^{-\gamma}\right)^p dx \\ &= n^{\gamma p - p - 1} + \left[n^{-p}\frac{x^{1-\gamma p}}{1-\gamma p}\right]_{\frac{1}{n}}^1 \\ &= \frac{\gamma p}{\gamma p - 1} n^{\gamma p - p - 1} - \frac{1}{\gamma p - 1} n^{-p} \end{aligned}$$

The second term always decays for large n. The first term grows if $\gamma > (p+1)/p$, decays if $\gamma < (p+1)/p$ and for the marginal case it is equal to the constant $(p+1)/p$. To deal with the case $\gamma p = 1$, we calculate:

$$\begin{aligned} ||f_n||_p^p &= \frac{1}{n}\left(n^{\gamma-1}\right)^p + \int_{\frac{1}{n}}^1 \left(\frac{1}{n}x^{-\gamma}\right)^p dx \\ &= n^{-p} + \left[n^{-p}\log x\right]_{\frac{1}{n}}^1 \\ &= n^{-p}(1 + \log n) \end{aligned}$$

which tends to zero for large n.

ii) To show that $\tau_{p'} \not\subset \tau_p$, it suffices to exhibit one element of $\tau_{p'}$ which is not in τ_p. Consider the set

$$U = \{f : ||f||_{p'} < 1\}$$

Clearly, we have that $U \in \tau_{p'}$. Suppose that $U \in \tau_p$. Then there must exist some $\epsilon > 0$ such that the set

$$B_\epsilon = \{f : ||f||_p < \epsilon\}$$

is contained in U, i.e., there is some ϵ such that $B_\epsilon \subset U$. Now, since $p' > p$, there exists some γ such that

$$\frac{p'+1}{p'} < \gamma < \frac{p+1}{p}.$$

Consider the sequence of f_n constructed above with such a γ. Clearly, we must have that there exists N such that $f_n \notin U$ for all $n > N$. On the other hand, there exists M such that $f_n \in B_\epsilon$ for all $n > M$, contradicting the assumption that $B_\epsilon \subset U$.

Exercise A.8. Verify that if (D, s) is a metric space, then the metric topology defines the same notions of convergence and continuity as the standard definitions for a metric space.

Exercise A.9. Let (X, τ) be a topological vector space

a) Show that if $(x_n)_{n=1}^{\infty}$ is a τ-Cauchy sequence, then $\{x_n\}_{n=1}^{\infty}$ is bounded.

b) Fix a local base $\dot{\beta}$. Show that a sequence $(x_n)_{n=1}^{\infty}$ is τ-Cauchy if and only if for any $B \in \dot{\beta}$ we can find an integer N such that

$$x_n - x_m \in B, \qquad \text{for all } n, m \geq N.$$

Solution a) Let W, be a neighbourhood of the origin. By part 1. of the proof of Lemma B.12, we can find $U \subset W$ a balanced open neighbourhood of the origin with $U+U \subset W$. Now, by the definition of a Cauchy sequence we can find N such that $x_m - x_n \in U'$ for all $n, m \geq N$. Since U' is balanced, this implies that $t^{-1}(x_n - x_m) \in U'$ for all $t > 1$. Moreover, since $\{x_1, \ldots, x_N\}$ is bounded, there exists $s > 1$ such that $s^{-1}x_m \in U'$ for $m \leq N$. For $n \geq N$, we have

$$s^{-1}x_n = s^{-1}(x_n - x_N) + s^{-1}x_N \in U' + U' \subset W.$$

Thus $x_n \in sW$ for $n \geq 0$ and we're done.

b) If (x_n) is τ-Cauchy, then since B is a neighbourhood of the origin, by the definition of τ-Cauchy there exists N such that

$$x_n - x_m \in B, \qquad \text{for all } n, m \geq N.$$

Now suppose that for any $B \in \dot{\beta}$ we can find an integer N such that

$$x_n - x_m \in B, \qquad \text{for all } n, m \geq N.$$

Let U be any neighbourhood of the origin. Since $\dot{\beta}$ is a local base, there exists $B \in \dot{\beta}$ with $B \subset U$. By the hypothesis, there exists N such that for all $n, n > N$ we have

$$x_n - x_m \in B \subset U.$$